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A NEW PRINCIPLE IN THE GEOMETRY OF NUMBERS, WITH SOME APPLICATIONS

BY

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1. Minkowski has discovered a geometrical principle which he applied with success to certain important problems in the theory of numbers. If we define *lattice-points* as those points in space of n dimensions whose (rectangular) coördinates are positive or negative integers or zero, his principle may be stated as follows:*

A surface in n-dimensional space, nowhere concave, possessing a center which coincides with one of the lattice-points of this n-space, and having a volume $\geqq 2^n$, will contain at least two more lattice-points, either inside the surface or upon its boundary.

He gave this theorem the following analytic form: Let $f(x_1, \cdots, x_n)$ be any function of $x_1, \cdots, x_n$, which vanishes when $x_1 = 0, \cdots, x_n = 0$, but possesses a definite positive value for any other system of values assigned to $x_1, \cdots, x_n$; let, moreover, the following functional equations be fulfilled:

$$\text{(I)} \qquad f(tx_1, \cdots, tx_n) = tf(x_1, \cdots, x_n), \quad \text{when} \quad t > 0,$$

$$\text{(II)} \qquad f(y_1 + z_1, \cdots, y_n + z_n) \leqq f(y_1, \cdots, y_n) + f(z_1, \cdots, z_n),$$

$$\text{(III)} \qquad f(-x_1, \cdots, -x_n) = f(x_1, \cdots, x_n).$$

Then the n-tuple integral $\int dx_1 \cdots dx_n$, taken over the region

$$f(x_1, \cdots, x_n) \leqq 1$$

in positive directions along the paths of integration, will possess a definite value J, and there exists at least one system of integers $l_1, \cdots, l_n$, such that

$$0 < f(l_1, \cdots, l_n) \leqq \frac{2}{\sqrt[n]{J}}.$$

2. Among the applications given by Minkowski, two will be mentioned here, and one later on (§ 12).

* *Geometrie der Zahlen*, Leipzig und Berlin, 1910, p. 76.

(α) Let

$$[f(x_1, \cdots, x_n)]^2 \equiv \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

be a positive, definite quadratic form in n variables, having a determinant D. Then the variables may be replaced by such integers $l_1, \cdots, l_n$ that we have*

$$0 < f^2 \leqq \frac{4}{\pi}\left[\Gamma\left(1+\frac{n}{2}\right)\right]^{2/n} D^{1/n}.$$

(β) Let

$$f(x_1, \cdots, x_n) \equiv |v_1| + \cdots + |v_n|,$$

where $v_1, \cdots, v_n$ are linear homogeneous forms in $x_1, \cdots, x_n$, with a non-vanishing determinant Δ. Let s pairs of these forms have conjugate imaginary coefficients. Then integers $l_1, \cdots, l_n$ exist such that we have, upon substitution,*

$$0 < f \leqq \left[\left(\frac{4}{\pi}\right)^s \Gamma(1+n) \cdot |\Delta|\right]^{1/n}.$$

3. A new geometrical principle will now be stated and proved. To make it somewhat more general in scope, a new definition of lattice-points will be necessary.

DEFINITION. Let the n-space defined by rectangular coördinates $x_1, \cdots, x_n$ be divided into equal rectangular spaces by the n systems of planes

$$x_1 = a_1 + b_1 t, \cdots, x_n = a_n + b_n t \quad (t = 0, \pm 1, \pm 2, \cdots),$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n$ are given real numbers. We shall call these spaces *fundamental parallelepipeds*. In each of them let there be located, in an arbitrary manner, a given number of points, say k; none of them, however, lying upon the boundaries of the parallelepipeds. *These points shall be designated lattice-points.*

It will be observed that this definition of lattice-points includes the former: set $a_i = \frac{1}{2}$, $b_i = 1$ $(i = 1, \cdots, n)$; and locate one lattice-point $(k = 1)$ at the center of each of the resulting parallelepipeds.

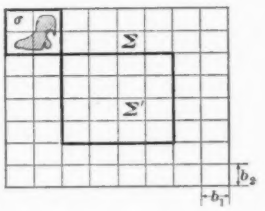
4. **Theorem I.** *Let S represent any limited open n-dimensional continuum in the n-space $x_1, \cdots, x_n$, having the (outer) volume V. By a suitable translation*

$$x_i' = x_i + \delta_i \qquad (i = 1, \cdots, n),$$

this continuum can be placed in such a position with reference to the fundamental parallelepipeds that the number of lattice-points L contained in the continuum or lying as near as we please to its boundary is greater than Vk/W, where W represents the volume, and k the number of lattice-points of a fundamental parallelepiped.

* *Geometrie der Zahlen*, p. 122.

Proof. We construct a parallelepiped σ whose edges parallel to the coördinate axes $X_1, \cdots, X_n$ are of lengths $Ab_1, \cdots, Ab_n$, A being a fixed integer (2 in the figure) taken so large that S can, by a translation, be placed entirely within σ. When so placed, the double surface obtained will be denoted by (S, σ). We then construct a parallelepiped Σ whose edges, parallel to the axes, are of lengths $Bb_1, \cdots, Bb_n$, B being a large (variable) integer (8 in the figure), which exceeds $2A$. We may assume that Σ contains just B^n fundamental parallelepipeds. Inside Σ we have a parallelepiped Σ' whose edges are of lengths $(B-2A)\,b_1, \cdots, (B-2A)\,b_n$, its faces being at distances $Ab_1, \cdots, Ab_n$ from those of Σ.



Now place (S, σ) in such a position inside Σ that one of the vertices of σ coincides with one of Σ, so that n of its faces lie upon n of the faces of Σ. From this position we obtain $\{(B-A)\,C+1\}^n$ different positions of (S, σ), all inside Σ, by means of translations of the form

$$x_i' = x_i + t_i \frac{b_i}{C} \qquad (i = 1, \cdots, n), \tag{1}$$

where t_i runs through all integral values from $-\infty$ to $+\infty$, independently for each subscript i, and where C represents a large positive integer.

5. Each of the $k\,(B-2A)^n$ lattice-points inside Σ' will lie inside or near to several of the surfaces S, if C be chosen large enough. We shall proceed to count these lattice-points in such a way that each is counted as often as it appears inside of, or as near as we please to a surface S. Call this number N.

Consider any one of these lattice-points P. To count the number of surfaces S inside of which it will appear, we may regard S as stationary and P as subjected to the translations (1). Then P will appear as vertices of parallelepipeds whose edges are of lengths $b_1/C, \cdots, b_n/C$. Let the least number of such parallelepipeds which entirely contain the continuum S be M_P. We may take C so large that all the vertices of these M_P parallelepipeds lie either within the continuum S, or as near as we please to its boundary. Summing for all of the lattice-points considered, we get

$$N > k\,(B-2A)^n\,M,$$

M being the smallest of the numbers M_P.

6. Since there are $\{(B-A)\,C+1\}^n$ different surfaces S inside Σ, it follows that there are

$$L > k\frac{(B-2A)^n\,M}{\{(B-A)\,C+1\}^n}$$

lattice-points inside or near the surface of at least one of them. If now B and C increase indefinitely, the right-hand member approaches

$$\lim_{C=\infty} k\frac{M}{C^n} = k\frac{V}{b_1 \cdots b_n} = \frac{k}{W}V$$

so that for a given positive ϵ we may take

$$L > \frac{k}{W}V - \epsilon.$$

Finally, since L is an integer, the inequality of the theorem follows unless Vk/W is an integer also. In this case however we consider first a continuum lying arbitrarily near to S and containing S within it. For this continuum of volume greater than V the desired inequality holds, and thus for S also. The theorem is therefore true.

7. When the different fundamental parallelepipeds are congruent with reference to the lattice-points contained, the statement in Theorem I reading "the number of lattice-points L contained in the continuum or lying as near as we please to limiting points of the continuum is greater than kV/W . . ." may be accentuated to read as follows: "*the number of lattice-points L contained in the continuum or belonging to limiting points of the continuum is greater than Vk/W. . . .*" The above proof may be modified to cover this statement. However, instead of doing this, we shall give a very elegant and independent proof of Theorem I, under the restrictions mentioned, furnished the author by Professor Birkhoff, which proof throws into evidence the sharper wording of the theorem. We shall limit ourselves to one lattice-point in each fundamental parallelepiped, although the proof, as given by Professor Birkhoff, is valid for any number. The spirit of the proof is shown equally well by limiting ourselves to space of two dimensions only.

Let us therefore assume the plane divided into a network R of equal rectangles, a single lattice-point A in each, similarly placed. Let a closed curve C be drawn, having an (outer) area V, the area of a rectangle being W.

We will superpose, by a translation, upon a single rectangle R_1 all those which are covered, in whole or in part, by V (its boundary included). It is then evident that a point P can be located in R_1 at which the superposed or adjoining portions of V are in number $L > V/W$, unless V/W = an integer and no part of the boundary of V appears inside R_1. But such a case could be avoided by subjecting C to a proper translation at the outset. In the reconstructed area V, P will appear as L points, congruent with reference to the rectangles R. Applying a translation such that these points coincide with L points A, the translated area V will contain in its interior or upon its boundary more than V/W lattice-points A.

8. **Extension of Theorem I.** *Let there be given a finite number of continua* $S_1, \cdots, S_m$, *overlapping or not, forming a network* $\bar{S}$, *and let* $\alpha_1, \cdots, \alpha_m$ *be arbitrary positive constants. Then by a suitable translation,* $\bar{S}$ *may be brought into such a position that, if* L_i *designates the number of lattice-points inside of or as near as we please to the surface of* S_i, *and* V_i *the outer volume of* S_i, *then* $\alpha_1 L_1 + \cdots + \alpha_m L_m > (\alpha_1 V_1 + \cdots + \alpha_m V_m)\, k / W$. The demonstration follows that of Theorem I, after S in the symbol (S, σ) is replaced by $\bar{S}$. Considering each continuum S_i in turn, we prove (using corresponding notation N_i, M_i):

$$\Sigma\alpha_i N_i > k\,(B - 2A)^n\, \Sigma\alpha_i M_i,$$

and then divide by $\{(B - 2A)\,C + 1\}^n$. Finally, $\Sigma\alpha_i L_i$ not being a continuous magnitude, the argument at the end of § 6 is valid here.

9. Minkowski's general analytical theorem (§ 1), so far as the last inequalities are concerned, follows immediately from Theorem I. Let the symbols $f(x_1, \cdots, x_n)$, J be defined as in Minkowski's theorem, and let S represent the continuum of points $(x_1, \cdots, x_n)$ satisfying the condition

$$f(x_1, \cdots x_n) < J^{-1/n}. \tag{2}$$

Then $V = 1$, and if "lattice-points" are defined as in § 1, we have $k/W = 1$ (cf. § 3). Applying Theorem I, we have $L \geqq 2$. Let $(y_1, \cdots, y_n)$ and $(-z_1, \cdots, -z_n)$ be two lattice-points inside S or upon its boundary (§ 7), in its new position. Set $y_1 + z_1 = l_1, \cdots, y_n + z_n = l_n$.

Now, if $(x_1', \cdots, x_n')$ be a lattice-point which after a translation lies in or on the boundary of the continuum defined by (2) and if $\delta_1, \cdots, \delta_n$ be the components of the translation, we have

$$f(x_1' - \delta_1, \cdots, x_n' - \delta_n) \leqq \frac{1}{J^{1/n}}.$$

Hence, we obtain

$$0 < f(l_1, \cdots, l_n) = f(y_1 + z_1, \cdots, y_n + z_n)$$

$$\leqq f(y_1 - \delta_1, \cdots, y_n - \delta_n) + f(-z_1 - \delta_1, \cdots, -z_n - \delta_n) \leqq 2J^{-1/n}.$$

10. **Quadratic forms.** Let F designate a positive definite quadratic form in n variables $x_1, \cdots, x_n$. We may write

$$F \equiv v_1^2 + \cdots + v_n^2,$$

where $v_1, \cdots, v_n$ are linear in the n variables and of determinant Δ. Then the determinant of F will be $D = \Delta^2$.

To apply the theorem in § 8, let us construct the network $\bar{S}$, consisting of the points respectively inside the surfaces $S_1, \cdots, S_m$; S_λ having for its

equation

$$F = (\lambda\phi)^{2/n}, \quad \text{where} \quad \phi = \frac{n+2}{2mJ};$$

here m is a large (variable) integer, and

$$J = \frac{\pi^{\frac{1}{2}n}}{\Delta\Gamma(1+\frac{1}{2}n)}$$

is the volume of $F = 1$. Let "lattice-points" be those points having integral coördinates, so that $k/W = 1$. Then, after a suitable translation,

$$\begin{aligned}(3)\qquad \alpha_1 L_1 + \cdots + \alpha_m L_m &> \alpha_1 V_1 + \cdots + \alpha_m V_m \\ &= \phi J(\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m).\end{aligned}$$

We shall put $\alpha_i = (i+1)^{2/n} - i^{2/n}$ when $i < m$,

$$\alpha_m = \frac{(n+2)g}{n\,m} - m^{2/n},$$

where $g = 1 + 2^{2/n} + \cdots + m^{2/n}$, and shall denote $L_i - L_{i-1}$ by p_i. Then (3) becomes

$$(4)\qquad \frac{(n+2)g}{n\,m} L_m - (p_1 + 2^{2/n} p_2 + \cdots + m^{2/n} p_m) > \frac{(n+2)g}{n\,m}.$$

Now, if $x_1', \cdots, x_n'$ be a lattice-point inside or near S_λ, and if we indicate the result of substituting $x_1' - \delta_1, \cdots, x_n' - \delta_n$ for $x_1, \cdots, x_n$ in v_i by v_i', then

$$(v_1')^2 + \cdots + (v_n')^2 < (\lambda\phi)^{2/n} + \epsilon,$$

where ϵ is a quantity small at will. If $x_1'', \cdots, x_n''$ be any other lattice-point, then the differences $v_1' - v_1''$, etc., are equal to the results obtained by substituting the integers $x_1' - x_1''$, etc., for x_1, etc., in $v_1, \cdots, v_n$. Hence

$$(5)\qquad (v_1' - v_1'')^2 + \cdots + (v_n' - v_n'')^2$$

will be a numerical value of F, for integral values of the variables involved.

Adding all the expressions (5) for every pair of lattice-points contained in the network $\overline{S}$, and denoting the sum $p_1 + p_2 + \cdots + p_m = L_m$ by P, we get

$$\begin{aligned}P\sum_{j=1}^{P}\{(v_1^{(j)})^2 + \cdots + (v_n^{(j)})^2\} - \left(\sum_{j=1}^{P} v_1^{(j)}\right)^2 - \cdots - \left(\sum_{j=1}^{P} v_n^{(j)}\right)^2 \\ < P\{p_1\phi^{2/n} + p_2(2\phi)^{2/n} + \cdots + p_m(m\phi)^{2/n}\} + \epsilon P^2 \\ < P\phi^{2/n}(P-1)\frac{(n+2)g}{n\,m} + \epsilon P^2,\end{aligned}$$

by (4). If we now divide by $\frac{1}{2}P(P-1)$, substitute the value of ϕ and pass to the limit $m=\infty$, we obtain

THEOREM II. *Let F be a positive definite quadratic form in n variables and of determinant D. Then integers $l_1, \cdots, l_n$, not all zero, may be substituted for the n variables such that the numerical value of F is not greater than*

$$\frac{2}{\pi}\left[\Gamma\left(1+\frac{n+2}{2}\right)\right]^{2/n} D^{1/n}.$$

It will be observed that the asymptotic value $nD^{1/n}/\pi e$ of this expression is one half that of Minkowski's limit (§ 2). Minkowski* has proved that the asymptotic value cannot fall below $nD^{1/n}/2\pi e$. The actual limit for $n=2$ was first determined by Hermite,† and for $n=3$, 4 and 5 by Korkine and Zolotareff.‡ The quotients of these limits divided by $D^{1/n}$ are respectively $2/\sqrt{3}, \sqrt[3]{2}, \sqrt{2}, \sqrt[5]{8}$.

11. **Linear forms.** Let $f \equiv |v_1| + \cdots + |v_n|$, where $v_1, \cdots, v_n$ are linear homogeneous forms in $x_1, \cdots, x_n$, of determinant $\Delta \neq 0$. If imaginary coefficients occur we assume that the forms having such coefficients appear in conjugate imaginary pairs. We can then utilize Theorem II. For, from the value of $|v_1|^2 + |v_2|^2$, the maximum value of $|v_1| + |v_2|$ is obtained by putting $|v_1| = |v_2|$. We find the following

THEOREM III. *Integers $l_1, \cdots, l_n$, not all zero, can be substituted for the variables $x_1, \cdots, x_n$ such that*

$$0 < |v_1| + \cdots + |v_n| \leqq \sqrt{\frac{2n}{\pi}}\left[\Gamma\left(1+\frac{n+2}{2}\right)\right]^{1/n} |\Delta|^{1/n}.$$

The asymptotic value

$$\frac{n}{\sqrt{\pi e}}|\Delta|^{1/n}$$

of this limit is smaller than that (§ 2) of Minkowski

$$\frac{n}{e}\left(\frac{4}{\pi}\right)^{s/n}|\Delta|^{1/n},$$

the more so the greater the number of pairs s of conjugate imaginary forms contained among $v_1, \cdots, v_n$. On the other hand, for low values of n, the limit given above is higher than that given by Minkowski.

*Journal für Mathematik, vol. 129 (1905), pp. 268–9.

†Journal für Mathematik, vol. 40 (1850), p. 263. The limits for $n=2$ and $n=3$ were virtually determined by Gauss and Seeber. Cf. Gauss' Werke, Bd. 1, p. 307, ff. and Bd. 2, p. 192, ff. Göttingen, 1876.

‡*Sur les formes quadratiques positives*, Mathematische Annalen, vol. 11 (1877), p. 242, ff.

12. **Approximation of irrational numbers by means of fractions.** Let $\alpha_1, \cdots, \alpha_{n-1}$ represent $n-1$ real positive numbers. The problem of finding $n-1$ rational fractions $x_1/z, \cdots, x_{n-1}/z$, having a common denominator, which are close approximations to the quantities α, has been solved by Hermite,* Kronecker,† and Minkowski,‡ and for $n=2$ by Hurwitz.§ We shall prove a theorem giving slightly closer limits, except for $n=2$, than those obtained by these writers.

Consider the linear forms

$$y_1 \equiv x_1 - \alpha_1 z, \quad y_2 \equiv x_2 - \alpha_2 z, \quad \cdots, \quad y_{n-1} \equiv x_{n-1} - \alpha_{n-1} z.$$

In the n-space $(y_1, \cdots, y_{n-1}, z)$ construct the lattice-points obtained by substituting all possible integers for $x_1, \cdots, x_{n-1}, z$. Construct also the planes

$$y_1 = a_1 + h_1, \quad \cdots, \quad y_{n-1} = a_{n-1} + h_{n-1}, \quad z = \tfrac{1}{2} + kh_n.$$

Here k is a given positive integer; $a_1, \cdots, a_{n-1}$ any set of $n-1$ numbers *not* representable by the forms $y_1, \cdots, y_{n-1}$; while $h_1, \cdots, h_{n-1}$ run, independently of each other, through all integers from $-\infty$ to $+\infty$. These planes divide the n-space into fundamental parallelepipeds, each of volume $W = k$, and each containing just k lattice-points. For there are just k sets of integers $x_1, \cdots, x_{n-1}, z$ which satisfy the inequalities

$$a_i + h_i < x_i - \alpha_i z < a_i + h_i + 1 \qquad (i = 1, \cdots, n-1),$$
$$\tfrac{1}{2} + kh_n < z < \tfrac{1}{2} + k(h_n + 1).$$

We now apply Theorem I, where S represents the continuum of points satisfying the conditions

$$|z| < a;$$

$$(6) \quad \left|\frac{y_i}{b}\right| + \left|\frac{2^n (n-1)^{n-1} z}{n^n a}\right| < 1 \quad \text{when} \quad \left|\frac{z}{a}\right| \leqq \left(\frac{n}{2(n-1)}\right)^{n-1},$$
$$\left|\frac{z}{a}\right| \left(\left|\frac{y_i}{b}\right| + 1\right)^{n-1} < 1 \quad \text{when} \quad 1 > \left|\frac{z}{a}\right| > \left(\frac{n}{2(n-1)}\right)^{n-1}$$
$$(i = 1, \cdots, n-1).$$

Here a and b are two positive real numbers subject to the restrictions

$$(7) \quad ab^{n-1} \left\{ \frac{n^{n-1}}{(n-1)^{n-1}} - \frac{(n-2)^n}{(n-1)^{n-1} n} + 2^n (n-1) \int_0^{1-2/n} \frac{v^{n-1}\, dv}{(1+v)^n} \right\} = 1, \quad b < \frac{1}{2}.$$

* Journal für Mathematik, vol. 40 (1850), p. 266.
† *Werke*, Bd. I, p. 636.
‡ *Geometrie der Zahlen*, p. 108 ff.
§ Mathematische Annalen, vol. 39 (1891), p. 279.

The volume of S is $V = 1$ by (7). We can therefore apply a translation

$$y_1' = y_1 + \delta_1, \quad \cdots, \quad y_{n-1}' = y_{n-1} + \delta_{n-1}, \quad z' = z + \delta_n$$

to S which will bring it into such a position that two lattice-points, $(y_{11}, \cdots, y_{1\,n-1}, z_1)$, $(y_{21}, \cdots, y_{2\,n-1}, z_2)$ are contained in S, or lie as near as we wish to its boundary.

The two integers z_1, z_2 cannot be equal. For otherwise

$$|x_{1i} - x_{2i}| = |y_{1i} - y_{2i}| = |(y_{1i} - \delta_i) - (y_{2i} - \delta_i)| < 2b + \epsilon < 1,$$

by (6) and (7), i. e., $x_{1i} - x_{2i} = 0$ for every subscript i, and hence the two lattice-points would not be distinct. If we therefore set

$$X_1 = x_{11} - x_{21}, \quad \cdots, \quad X_{n-1} = x_{1\,n-1} - x_{2\,n-1}, \quad Z = z_1 - z_2,$$

we have $Z \geqq 1$, and we can prove that

$$|X_i - \alpha_i Z| < 2b + \epsilon_1, \qquad |X_i - \alpha_i Z|\,|Z|^{1/n-1} < a^{1/n-1} b + \epsilon_2,$$

where ϵ_1 and ϵ_2 are quantities as small as we please. If now we notice that from (7)

$$ab^{n-1}\left(\frac{n}{n-1}\right)^{n-1}\left[1 + \left(\frac{n-2}{n}\right)^{n-2}\right] \leqq 1,$$

and that z may be assumed positive, these results give at once:

THEOREM IV. *Given $n-1$ positive quantities $\alpha_1, \cdots, \alpha_{n-1}$, and an arbitrarily small quantity $b < \frac{1}{2}$, we can find n integers $X_1, \cdots, X_{n-1}, Z$ such that the $n-1$ differences*

$$\left|\frac{X_i}{Z} - \alpha_i\right|$$

are not greater than $2b$, and at the same time are all not greater than

$$\frac{\gamma}{Z^{n/n-1}} = \frac{(n-1)\,Z^{-(n/n-1)}}{n\left[1 + \left(\frac{n-2}{n}\right)^{n+2}\right]^{1/n-1}}.$$

The common denominator Z need not be taken greater than $2a \leqq 2\,(\gamma / b)^{n-1}$.

For $n = 2$, Hurwitz proved (l. c.) that $\gamma = 1/\sqrt{5}$, and Minkowski* proved that, for $n \geqq 2$, $\gamma < (n-1)/n$. For large values of n, this may be written $\gamma = e^{-1/n}$, while the expression for γ in Theorem IV becomes $(e + 1/e)^{-1/n}$.

* *Geometrie der Zahlen*, p. 112.

AN APPLICATION OF SEVERI'S THEORY OF A BASIS TO THE KUMMER AND WEDDLE SURFACES*

BY

F. R. SHARPE AND C. F. CRAIG

In a series of papers F. Severi† has developed a theory by means of which any linear system of algebraic curves on an algebraic surface can be expressed linearly in terms of a finite number of such systems. He has also applied this theory to the study of the curves on the Fano quartic surface and to the determination of the group of birational transformations which leave this surface invariant.

In the following paper this method is applied to the study of certain of the birational transformations which leave the Kummer and Weddle surfaces invariant. In particular the question of the periodicity of the product of two such transformations is studied. Some new relations among the different types of transformation are obtained.

1. The Kummer Surface

Let A, B, E, F be any Göpel even tetrad of nodes on the general Kummer surface K_4. The notation‡

$$(1) \qquad \begin{matrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{matrix}$$

will be used to denote the complete configuration of the sixteen nodes on K_4. Variables x, y, z, w will be so chosen that $A \equiv (1,0,0,0), B \equiv (0,1,0,0), E \equiv (0,0,1,0)$ and $F \equiv (0,0,0,1)$.

* Presented to the Society February 28, 1914.

† *Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica*, Mathematische Annalen, vol. 62 (1906), pp. 194–225; *La base minima pour la totalité des courbes tracées sur une surface algébrique*, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), pp. 449–468; *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

‡ See Hudson, *Kummer's Quartic Surface*, Chapters I, II, and VII.

The surface K_4, referred to this system of coördinates, is invariant under the cubic inversion*

$$x' : y' : z' : w' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{w}. \tag{2}$$

Under this transformation the planes of each pencil through an edge of the tetraedron of inversion are interchanged in pairs. Through each pair of vertices are two six-point conics, namely:

$$\begin{Bmatrix} ABDGKO, \\ ABCHLP, \end{Bmatrix} \quad \begin{Bmatrix} AEJKLM, \\ AEIONP, \end{Bmatrix} \quad \begin{Bmatrix} AFCDJN, \\ AFGHIM, \end{Bmatrix}$$

$$\begin{Bmatrix} EFCHKO, \\ EFDGLP, \end{Bmatrix} \quad \begin{Bmatrix} BFIKLN, \\ BFJMOP, \end{Bmatrix} \quad \begin{Bmatrix} BECDIM, \\ BEGHJN. \end{Bmatrix}$$

Hence these pairs of conics are interchanged under (2). This requires that twelve nodes interchange in pairs as follows

$$(CG)\,(DH)\,(IJ)\,(KP)\,(LO)\,(MN), \tag{3}$$

where (CG) indicates the interchange of the points C and G. It follows that each of the four remaining conics $CGIJLO$, $DHLOMN$, $DHIJKP$, and $CGKPMN$, which are plane sections of K_4 by the same quadric, is invariant under (2). Since K_4 is invariant and each of the vertices A, B, E, F is transformed into the opposite face of the tetraedron of inversion by (2), the image on K_4 of each vertex is the rational plane section of the surface by the opposite face. Denote a general plane section of K_4 by C_4 and use the symbol $\sim$ to denote "is transformed into." The transformations of the vertices are

$$\begin{aligned} A &\sim C_4 - B - E - F, \qquad B \sim C_4 - A - E - F, \\ E &\sim C_4 - A - B - E, \qquad F \sim C_4 - A - B - E, \end{aligned} \tag{4}$$

respectively.† By (2) any plane is transformed into a cubic surface which contains the edges of the tetraedron $ABEF$. This surface intersects K_4 in C_{12}, a curve of order 12. Since genus is invariant under birational transformation and C_{12} is the image of any plane section of K_4, the genus of this curve is 3. The inversion is also symmetric with respect to all the vertices: hence

$$C_4 \sim 3C_4 - 2(A + B + E + F). \tag{5}$$

* See J. I. Hutchinson, *On some birational transformations of the Kummer surface into itself*, Bulletin of the American Mathematical Society, ser. 2, vol. 7 (1901), pp. 211–217; see p. 212.

† The symbol $\lambda C_4 - \alpha A - \beta B - \cdots$ denotes the complete curve of intersection of the Kummer surface with a surface of order λ. This curve of intersection passes through the points A, B, $\cdots$, 2α, 2β, $\cdots$, times respectively, is of order 4λ, and is of genus $1 + 2\lambda^2 - \alpha^2 - \beta^2 - \cdots$. For details as to the meaning of symbolic addition and subtraction see Picard et Simart, *Fonctions algébriques de deux variables*, vol. 2, pp. 104–116 or the papers of Severi already cited.

The partial transformations (3), (4), and (5) completely* express the transformation of the linear systems of the curves on K_4 under the inversion (2). This complete transformation will be denoted by $\left[\begin{smallmatrix} AB \\ EF \end{smallmatrix}\right]$. The inversion as thus expressed is geometric in character and independent of the coördinate system.

The inversion with respect to any other Göpel even tetrad of nodes can readily be written. The transformation $\left[\begin{smallmatrix} AB \\ IJ \end{smallmatrix}\right]$ is

$$(CK)\,(DL)\,(EF)\,(GP)\,(HO)\,(MN),$$

$$A \sim C_4 - B - I - J, \quad B \sim C_4 - A - I - J, \quad I \sim C_4 - A - B - J,$$

$$J \sim C_4 - A - B - I, \quad C_4 \sim 3C_4 - 2(A + B + I + J);$$

and $\left[\begin{smallmatrix} AB \\ MN \end{smallmatrix}\right]$ is

$$(CO)\,(DP)\,(EF)\,(GL)\,(HK)\,(IJ),$$

$$A \sim C_4 - B - M - N, \quad B \sim C_4 - A - M - N,$$

$$M \sim C_4 - A - B - N, \quad N \sim C_4 - A - B - M,$$

$$C_4 \sim 3C_4 - 2(A + B + M + N).$$

By the symbol $\left[\begin{smallmatrix} AB \\ EF \end{smallmatrix}\right]\left[\begin{smallmatrix} AB \\ IJ \end{smallmatrix}\right]$, referred to as a product, is meant that a basis (α) of elements is transformed into a configuration (β) by $\left[\begin{smallmatrix} AB \\ EF \end{smallmatrix}\right]$ and that (β) is transformed into (γ) by $\left[\begin{smallmatrix} AB \\ IJ \end{smallmatrix}\right]$. The product transforms (α) into (γ). Thus $\left[\begin{smallmatrix} AB \\ EF \end{smallmatrix}\right]\left[\begin{smallmatrix} AB \\ IJ \end{smallmatrix}\right]$ is

$$A \sim 2C_4 - A - 2B - E - F - I - J,$$

$$B \sim 2C_4 - 2A - B - E - F - I - J,$$

$$E \sim C_4 - A - B - E, \quad F \sim C_4 - A - B - F, \quad I \sim C_4 - A - B - I,$$

$$J \sim C_4 - A - B - J, \quad C_4 \sim 5C_4 - 4(A + B) - 2(E + F + I + J),$$

$$(CP)\,(DO)\,(GK)\,(HL)\,(MM)\,(NN).$$

The symmetry of this result in E, F, I, and J shows that the transformation is of period two.† Similarly the products $\left[\begin{smallmatrix} AB \\ EF \end{smallmatrix}\right]\left[\begin{smallmatrix} AB \\ MN \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} AB \\ IJ \end{smallmatrix}\right]\left[\begin{smallmatrix} AB \\ MN \end{smallmatrix}\right]$ and

* See Severi, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), p. 465.

† See H. F. Baker, *Note on some transformations of a general Kummer surface*, Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912–13), pp. 302–312, where this result is obtained by a different method. The operations in Baker's paper may be identified with those of the present paper by the following key:

$$\omega \equiv \left[\begin{smallmatrix} AB \\ EF \end{smallmatrix}\right], \quad \omega U_{DA} \equiv \left[\begin{smallmatrix} AB \\ IJ \end{smallmatrix}\right], \quad \omega V_{DA} \equiv \left[\begin{smallmatrix} AB \\ MN \end{smallmatrix}\right],$$

and

$$\vartheta_2\,\vartheta_3, \quad \vartheta_3\,\vartheta_1, \quad \vartheta_1\,\vartheta_2$$

with

$$(AF)\,(BE)\,(CH)\,(DG)\,(IN)\,(JM)\,(KP)\,(LO),$$
$$(AE)\,(BF)\,(CG)\,(DH)\,(IM)\,(JN)\,(KO)\,(LP),$$
$$(AB)\,(CD)\,(EF)\,(GH)\,(IJ)\,(KL)\,(MN)\,(OP),$$

C_4 remaining invariant under each collineation.

$\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\IJ\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\MN\end{smallmatrix}\right]$ are each of period two. Hence *three cubic inversions, taken with respect to Göpel even tetraedra which have an edge in common, generate an abelian group of order eight and type* (1, 1, 1). There are 120 such edges and hence 120 such groups.

The tetraedron $ACIK$ has a vertex in common with the tetraedron $ABEF$. The product $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}AC\\IK\end{smallmatrix}\right]$ is

$$
\begin{aligned}
&A \sim 3C_4 - 2A - 2(C+I+K) - (G+J+P),\\
&B \sim 2C_4 - 2A - (C+I+K) - (G+P),\\
&E \sim 2C_4 - 2A - (C+I+K) - (J+P),\\
&F \sim 2C_4 - 2A - (C+I+K) - (G+J),\\
&G \sim C_4 - A - (I+K), \qquad C \sim E, \qquad (6)\\
&J \sim C_4 - A - (C+K), \qquad I \sim B,\\
&P \sim C_4 - A - (C+I), \qquad K \sim F,\\
&C_4 \sim 7C_4 - 6A - 4(C+I+K) - 2(G+J+P),\\
&\qquad (DNO)\,(HLM),
\end{aligned}
$$

where (DNO) is the cyclic substitution on its elements. If we write

$$\alpha = C+I+K, \qquad \beta = G+J+P, \qquad \gamma = B+E+F,$$

then (6) in part becomes

$$
\begin{aligned}
&C_4 \sim 7C_4 - 6A - 4\alpha - 2\beta, \qquad A \sim 3C_4 - 2A - 2\alpha - \beta, \qquad \alpha \sim \gamma, \qquad (7)\\
&\beta \sim 3C_4 - 3A - 2\alpha, \qquad \gamma \sim 6C_4 - 6A - 3\alpha - 2\beta.
\end{aligned}
$$

If (7) is non-periodic, (6) can not be periodic. Transform (7) through

$$X = C_4 - 2A, \qquad Y = A - \beta, \qquad Z = \alpha - 2\beta + \gamma,$$
$$W = A - \alpha, \qquad T = \alpha,$$

and it becomes

$$
\begin{aligned}
&X \sim X, \qquad Y \sim Y, \qquad Z \sim Z, \qquad W \sim 3X + 3Y - Z + W,\\
&T \sim -2Y + Z + 2W + T,
\end{aligned}
$$

which is clearly non-periodic. Hence (6) is non-periodic. By suitable choice of variables the periodicity of the transformation given by the product of any two inversions, taken with respect to Göpel even tetraedra which have one node in common, can always be made to depend on the periodicity of (7). Hence the products of this type are all non-periodic.

Consider next the product $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}IJ\\MN\end{smallmatrix}\right]$ by which

$$
\begin{gathered}
A \approx 3C_4 - (A+E+F) - 2(I+J+M+N), \\
B \approx 3C_4 - (B+E+F) - 2(I+J+M+N), \\
E \approx 3C_4 - (A+B+E) - 2(I+J+M+N), \\
F \approx 3C_4 - (A+B+F) - 2(I+J+M+N), \\
I \approx C_4 - (I+M+N), \qquad J \approx C_4 - (J+M+N), \\
M \approx C_4 - (I+J+M), \qquad N \approx C_4 - (I+J+N), \\
C_4 \approx 9C_4 - 2(A+B+E+F) - 6(I+J+M+N), \\
(CD)\,(GH)\,(KL)\,(OP).
\end{gathered} \tag{8}
$$

The periodicity of (8) depends on that of

$$C_4 \approx 9C_4 - 2\alpha - 6\beta, \qquad \alpha \approx 12C_4 - 3\alpha - 8\beta, \qquad \beta \approx 4C_4 - 3\beta,$$

where $\alpha = A+B+E+F$ and $\beta = I+J+M+N$. Transform this through

$$X = C_4 - \beta, \qquad Y = 2C_4 - \alpha - \beta, \qquad Z = \beta,$$

and it becomes

$$Y \approx Y, \qquad X \approx 2Y + X, \qquad Z \approx 4X + Z,$$

which is non-periodic. Hence (8) is non-periodic. It is to be noticed that the four vertices of each tetraedron of this product are invariant as a whole under the inversion taken with respect to the other tetraedron. The non-periodicity of the product of two inversions, taken with respect to any similarly chosen pair of Göpel even tetraedra, is an immediate consequence.

In a similar manner the transformation $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}JK\\NO\end{smallmatrix}\right]$ may be shown to be non-periodic. The distinction between this case and that immediately preceding is that $J\,K\,N\,O$ are not invariant as a whole under $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]$ and $A\,B\,E\,F$ are not invariant as a whole under $\left[\begin{smallmatrix}JK\\NO\end{smallmatrix}\right]$. The non-periodicity of all similar products follows.

This completes the analysis of the product of two inversions taken with respect to tetraedra chosen in all possible positions. Hence follows the

THEOREM. *The product of any two cubic inversions of the general Kummer surface is periodic only if the Göpel even tetraedra, with respect to which the respective inversions are taken, have an edge in common.*

An immediate consequence of this is the theorem of Hutchinson* that the sixty inversions generate a group of infinite order.

* Loc. cit., p. 212.

Any line through a node of K_4 meets the surface in two other points. The transformation which interchanges these points is a monoidal cubic involution having the node for fundamental point. Under this nodal projection any plane section of K_4 is transformed into a C_{12} of genus three. Hence for the node A

$$C_4 \sim 3C_4 - 4A. \tag{9}$$

The image of the node is the intersection of the surface K_4 with the tangent cone at the node, namely a C_8 of genus zero. Hence

$$A \sim 2C_4 - 3A. \tag{10}$$

The other nodes remain invariant. Equations (9) and (10) are all that will be written to denote the complete nodal projection and will be indicated by $[A]$.

The product of two nodal projections $[B][A]$ gives

$$C_4 \sim 9C_4 - 12A - 4B, \quad A \sim 2C_4 - 3A, \quad B \sim 6C_4 - 8A - 3B,$$

the rest of the nodes being invariant. This transformation may be shown to be non-periodic by the methods employed above.*

If a point be transformed by a cubic inversion and its image transformed by a nodal projection, the resulting transformation is either involutorial or non-periodic. For $\left[\begin{smallmatrix} AB \\ EF \end{smallmatrix}\right][A]$ gives

$$C_4 \sim 5C_4 - 6A - 2(B+E+F), \quad A \sim 3C_4 - 4A - (B+E+F),$$
$$B \sim C_4 - A - (E+F), \quad E \sim C_4 - A - (B+F),$$
$$F \sim C_4 - A - (B+F), \quad (CG)\,(DH)\,(IJ)\,(KP)\,(LO)\,(MN).$$

This is easily verified to be involutorial. If the node of projection is not a vertex of the tetraedron of inversion, every such product is of the same type as $\left[\begin{smallmatrix} EF \\ AB \end{smallmatrix}\right][C]$, under which

$$C_4 \sim 9C_4 - 12C - 2(A+B+E+F),$$
$$A \sim 3C_4 - 4C - (B+E+F), \quad B \sim 3C_4 - 4C - (A+E+F),$$
$$E \sim 3C_4 - 4C - (A+B+F), \quad F \sim 3C_4 - 4C - (A+B+E),$$
$$G \sim 2C_4 - 3C, \quad C \sim G, \quad (DH)\,(IJ)\,(KP)\,(LO)\,(MN).$$

The non-periodicity of this transformation may be established as above. Hence *the product of a cubic inversion and a nodal projection is non-periodic except when the fundamental point of the projection is one of the basis points of the inversion.*

* For another proof see V. Snyder, *An application of a* (1, 2) *quaternary correspondence to the Weddle and Kummer surfaces*, these Transactions, vol. 12 (1911), pp. 354–366; in particular, see p. 364.

2. The Weddle Surface

The Weddle surface W_4 can be obtained from the Kummer surface by birational transformation.* Under this transformation one node of the Kummer surface is transformed into a cubic curve on W_4. The six conics of K_4 through this node become the six nodes on the cubic curve on W_4. The remaining ten conics on K_4 are transformed into the ten lines of intersection of the pairs of planes determined by the six nodes on W_4. The other fifteen nodes of K_4 become the fifteen connecting lines of the six nodes of W_4. A plane section of K_4 is transformed into a curve of order eight on W_4.

In the diagram (1), if the cubic curve is taken as A, the residual elements will then be the fifteen lines joining the pairs of nodes on W_4. The six nodes will be denoted by b, c, d, e, i, and m; the remaining ten lines will be $a, f, g, h, j, k, l, n, o$, and p. These may be identified by the conics on K_4 to which they correspond. For example, through the node d pass the lines B, C, H, L, P, and the cubic curve A; the line k meets the lines C, G, I, J, L, and O.

Let variables x, y, z, w be so chosen that

$$e \equiv (1,0,0,0), \qquad i \equiv (0,1,0,0), \qquad m \equiv (0,0,1,0),$$

$$b \equiv (0,0,0,1), \qquad c \equiv (a,b,c,d), \qquad d \equiv (1/a, 1/b, 1/c, 1/d).$$

The equation of the Weddle surface, written in terms of these coördinates is (18) and is invariant under the transformation† (2). Under this transformation the pencils of planes through the edges of the tetraedron and the nodes c and d are interchanged in pairs. Hence the lines joining each vertex to c and d are interchanged. Thus

$$(CD)\,(GH)\,(KL)\,(OP). \tag{12}$$

Under (2) an edge of the tetraedron of inversion is transformed into the opposite edge, or

$$(EF)\,(IJ)\,(MN). \tag{13}$$

Further, the cubic curve A, through the six nodes, has the line B as image and conversely, thus

$$(AB). \tag{14}$$

A quadric surface through the six nodes intersects W_4 in a curve C_8 of order eight which has a double point at each node and the lines $a, f, g, h, j, k, l, n, o$, and p for bisecants. Under (2) this curve is transformed into another

* The transformation here used is employed by Snyder, loc. cit., p. 360 and is the dual of the correspondence given by Hudson, loc. cit., p. 169.

† See Hutchinson, loc. cit., p. 216.

curve of the same system, since the quadric surface is transformed into another one through the six nodes; hence

$$(C_8\, C_8). \tag{15}$$

It follows from (12), (13), (14), and (15) that the complete inversion is

$$(C_8\, C_8)\,(AB)\,(CD)\,(EF)\,(GH)\,(IJ)\,(KL)\,(MN)\,(OP).$$

Since C_8 on W_4 corresponds to C_4 on K_4, it follows that this is a collineation on K_4. Hence *the collineation group on K_4 corresponds to the inversion group on W_4. On W_4 the product of two inversions is an inversion* and the inversion group is of order* 16.

Any line through a node of W_4 meets the surface in two other points. The transformation which interchanges these points is involutorial. Consider the projection N_c from the node c. The tangent cone at the node meets W_4 in a curve of order eight, which consists of the lines B, D, G, K, and O and the cubi A, hence (Ac). A plane through B intersects W_4 in B and a cubic curve. This curve intersects B in c, d, and a residual point R which is the image of d in this plane. As the plane rotates, R describes the line B. Hence (Bd), and similarly for the four remaining lines through c. Under N_c the line M is transformed into the ine of intersection of the planes Mc and Nd, and conversely. Hence (Mo), and similarly for all remaining lines. The collected result is

$$\begin{aligned}&(Ac)\,(Bd)\,(Ca)\,(Db)\,(Eg)\,(Fh)\,(Ge)\,(Hf)\\ &(Ik)\,(Jl)\,(Ki)\,(Lj)\,(Mo)\,(Np)\,(Om)\,(Pn).\end{aligned} \tag{16}$$

Since $C_8 = 2a + B + C + D + E + I + M$, it follows from (16) that $C_8 \sim 2C + d + a + b + g + k + o$. By writing the symbolic sums for C_8 which contain the elements a, b, d, g, k, o, these may be expressed in terms of C_8 and the elements $A, B, \cdots, P$. Hence

$$C_8 \sim 3C_8 - (A + B + \cdots + P). \tag{17}$$

The transformation N_c is given by (16) and (17). In like manner the transformation N_d is given by (17) and

$$\begin{aligned}&(Ad)\,(Bc)\,(Cb)\,(Da)\,(Eh)\,(Fg)\,(Gf)\,(He)\\ &(Il)\,(Jk)\,(Kj)\,(Li)\,(Mp)\,(No)\,(On)\,(Pm).\end{aligned}$$

* In Hutchinson's paper, loc. cit., p. 217, the product of two inversions is stated to be of infinite order. The formulas used for changing the fundamental tetraedron from $e\ i\ m\ b$ to $e\ i\ m\ c$ should read, in his notation,

$$w_1 = w, \qquad \sqrt{b^2 - a^2}\, x_1 = bw - ax, \qquad \sqrt{c^2 - a^2}\, y_1 = cw - ay, \qquad \sqrt{d^2 - a^2}\, z_1 = dw - az.$$

With this change it is seen that the product of the two is an inversion.

The product of these transformations is the inversion with respect to the remaining nodes. Hence *the cubic inversion with respect to any tetraedron of nodes on the Weddle surface is the product of the nodal projections with respect to the two residual nodes.**

The points (x, y, z, w), $(1/x, 1/y, 1/z, 1/w)$, (a, b, c, d), and $(1/a, 1/b, 1/c, 1/d)$ are coplanar if

$$\begin{vmatrix} x & y & z & w \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} & \frac{1}{w} \\ a & b & c & d \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \end{vmatrix} = 0. \tag{18}$$

This condition is satisfied if (x, y, z, w) is on the Weddle surface. In other words, a point on W_4 and its image under a cubic inversion are coplanar with the residual nodes of the tetraedron of inversion. Thus any plane through the residual nodes is invariant under an inversion, as is well known for the equivalent product of the two nodal projections.

Let the nodal projections be designated by N_i, $i = 1, 2, \cdots, 6$. The inversions will be $N_i N_k = I_{ik}$. Hence $N_i = I_{ik} N_k$. There results immediately the known theorem that the group of the nodal projections is of order 32. For it is the direct product of the inversion group of order 16 by the group generated by any nodal projection.

CORNELL UNIVERSITY,
January, 1914.

* The algebraical equivalent of this theorem is given by Baker, *Multiply Periodic Functions*, p. 154.

TRANSFORMATIONS OF SURFACES OF VOSS*

BY

LUTHER PFAHLER EISENHART

INTRODUCTION

Voss† was the first to study surfaces which are characterized by the property of containing two families of geodesics which form a conjugate system. These surfaces belong to the class admitting a continuous deformation in which a conjugate system remains conjugate; for a *surface of Voss* it is the geodesic conjugate system which possesses this property. These surfaces play an important rôle also in the determination of congruences whose developables meet the focal surfaces in their lines of curvature. It is the purpose of this paper to establish transformations of surfaces of Voss into surfaces of the same kind, and to study some of the geometrical properties of these transformations.

The geodesic conjugate system of a surface‡ V has the same spherical representation as the asymptotic lines of a pseudospherical surface P. Moreover, the determination of all surfaces V with the spherical representation of a given surface P requires the integration of an equation of Laplace with equal invariants. We shall say that each of these surfaces is *conjugate* to P. In § 1 we recall the formulas defining a pseudospherical surface P, the formulas for a conjugate surface V and the equations of a Bäcklund transformation of P into another pseudospherical surface P_1. Such a transformation involves a constant σ (the angle between the tangent planes to P and P_1) and a function θ whose determination requires the solution of a Riccati equation. When a surface V is given, each pair of quantities θ and σ determine a new surface V_1 such that the developables of the congruence of lines joining corresponding points on V and V_1 meet these surfaces in geodesic conjugate systems. Moreover, V_1 is a conjugate of the surface P_1 which is determined by the Bäcklund transformation of P by means of θ and σ. In order to put in evidence the functions we say that V_1 is obtained from V by a transformation $\Omega(\theta, \sigma)$.

* Presented to the Society, December 31, 1913.

† *Ueber diejenigen Flächen auf denen zwei Scharen geodätischer Linien ein conjugirtes System bilden*, Sitzungsberichte der K. Akademie zu München (1888), pp. 95–102.

‡ Throughout the paper we shall denote by V a surface of Voss.

Each surface V conjugate to P is transformable by the same pair of functions θ and σ. However, for a given pair (θ, σ) there exists a surface V such that the lines joining its points to the corresponding points on the surface V_1 resulting from the transformation $\Omega(\theta, \sigma)$ are concurrent. By means of certain additional functions connected with this special case we are able to show that the transformations $\Omega(\theta, \sigma)$ are of the Moutard type for equations with equal invariants. In a subsequent paper we shall show that there exist transformations of general conjugate systems with equal tangential invariants — transformations which are, in certain respects, generalizations of the transformations $\Omega(\theta, \sigma)$.

The transformations $\Omega(\theta, \sigma)$ admit a "theorem of permutability" (cf. § 6). The remainder of the paper deals with the congruence of lines joining corresponding points on V and V_1, and also the other congruence formed by the lines of intersection of the tangent planes at corresponding points on V and V_1; the latter is a normal congruence.

1. Equations of a Surface V and Preliminary Formulas

We consider a surface V referred to the geodesic conjugate system. Since this system has the same spherical representation as the asymptotic lines on a pseudospherical surface P, the linear element of this spherical representation can be given the form

$$d\sigma^2 = du^2 + 2\cos 2\omega\, du dv + dv^2, \tag{1}$$

where ω is a function of u and v satisfying the equation

$$\frac{\partial^2 \omega}{\partial u \partial v} + \sin\omega\cos\omega = 0.^* \tag{2}$$

If X_1, Y_1, Z_1 and X_2, Y_2, Z_2, denote respectively the direction-cosines of the bisectors of the angles between the parametric curves of the spherical representation, and X, Y, Z the direction-cosines of the normal to a surface V with this representation, we have

$$\begin{aligned}
&\frac{\partial X_1}{\partial u} = -\frac{\partial \omega}{\partial u} X_2 - \sin\omega\, X, && \frac{\partial X_1}{\partial v} = \frac{\partial \omega}{\partial v} X_2 + \sin\omega\, X,\\
&\frac{\partial X_2}{\partial u} = \frac{\partial \omega}{\partial u} X_1 - \cos\omega\, X, && \frac{\partial X_2}{\partial v} = -\frac{\partial \omega}{\partial v} X_1 - \cos\omega\, X,\\
&\frac{\partial X}{\partial u} = \sin\omega\, X_1 + \cos\omega\, X_2, && \frac{\partial X}{\partial v} = -\sin\omega\, X_1 + \cos\omega\, X_2.^\dagger
\end{aligned} \tag{3}$$

* E., p. 289. A reference of this kind is to the author's *Treatise on the differential geometry of curves and surfaces*, Ginn and Company, Boston (1909).

† Cf. Bianchi, *Lezioni di geometria differenziale*, vol. 1, p. 320, Pisa (1902).

The equation satisfied by the tangential coördinates X, Y, Z, and W of the surface V is

$$\frac{\partial^2 \phi}{\partial u \partial v} + \cos 2\omega\, \phi = 0.* \tag{4}$$

In terms of the tangential coördinates the rectangular coördinates are of the form

$$x = WX + \frac{1}{\sin 2\omega}\left[X_1 \cos\omega\left(\frac{\partial W}{\partial u} - \frac{\partial W}{\partial v}\right) + X_2 \sin\omega\left(\frac{\partial W}{\partial u} + \frac{\partial W}{\partial v}\right)\right].\dagger \tag{5}$$

From these we obtain

$$\begin{aligned} \frac{\partial x}{\partial u} &= -\frac{D}{\sin 2\omega}(\cos\omega\, X_1 + \sin\omega\, X_2), \\ \frac{\partial x}{\partial v} &= \frac{D''}{\sin 2\omega}(\cos\omega\, X_1 - \sin\omega\, X_2), \end{aligned} \tag{6}$$

where

$$\begin{aligned} D &= -\left(\frac{\partial^2 W}{\partial u^2} - 2\cot 2\omega \frac{\partial\omega}{\partial u}\frac{\partial W}{\partial u} + \frac{2}{\sin 2\omega}\frac{\partial\omega}{\partial u}\frac{\partial W}{\partial v} + W\right), \\ D'' &= -\left(\frac{\partial^2 W}{\partial v^2} + \frac{2}{\sin 2\omega}\frac{\partial\omega}{\partial v}\frac{\partial W}{\partial u} - 2\cot 2\omega \frac{\partial\omega}{\partial v}\frac{\partial W}{\partial v} + W\right).\ddagger \end{aligned} \tag{7}$$

The Codazzi equations for the surface V may be given the form

$$\frac{\partial D}{\partial v} - \frac{2}{\sin 2\omega}\frac{\partial\omega}{\partial u}D'' = 0, \qquad \frac{\partial D''}{\partial u} - \frac{2}{\sin 2\omega}\frac{\partial\omega}{\partial v}D = 0.\S \tag{8}$$

When the pseudospherical surface P with the representation (1) of its asymptotic lines is subjected to a Bäcklund transformation, the linear element of the spherical representation of the asymptotic lines on the new surface P_1 is given by

$$d\sigma_1^2 = du^2 + 2\cos 2\theta\, du dv + dv^2, \tag{9}$$

where θ is a solution of equation (2) given by

$$\begin{aligned} \sin\sigma\left(\frac{\partial\theta}{\partial u} - \frac{\partial\omega}{\partial u}\right) + (\cos\sigma + 1)\sin(\theta + \omega) &= 0, \\ \sin\sigma\left(\frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial v}\right) + (\cos\sigma - 1)\sin(\theta - \omega) &= 0, \end{aligned} \tag{10}$$

* E., p. 415.
† E., p. 163 with the aid of equations (3).
‡ Cf. E., p. 164.
§ E., p. 200. Equations (36).

σ being the constant angle between the tangent planes to P and P_1 at corresponding points.* The direction-cosines X', Y', and Z' of the normal to P_1 are of the form

$$X' = \cos\sigma\, X + \sin\sigma\,(\sin\theta\, X_1 - \cos\theta\, X_2)\,, \tag{11}$$

and the direction-cosines $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ of the line joining corresponding points of P and P_1, and tangent to these surfaces at these points, are of the form

$$\bar{X} = \cos\theta\, X_1 + \sin\theta\, X_2\,. \tag{12}$$

From (11) we obtain with the aid of equations (3)

$$\frac{\partial X'}{\partial u} = \sin\theta\, X_1' + \cos\theta\, X_2', \qquad \frac{\partial X'}{\partial v} = -\sin\theta\, X_1' + \cos\theta\, X_2', \tag{13}$$

where we have put

$$\begin{aligned} X_1' &= \sin\omega\,\big(\cos\sigma\,(\sin\theta\, X_1 - \cos\theta\, X_2) - \sin\sigma\, X\big) \\ &\qquad - \cos\omega\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,, \\ X_2' &= -\cos\omega\,\big(\cos\sigma\,(\sin\theta\, X_1 - \cos\theta\, X_2) - \sin\sigma\, X\big) \\ &\qquad - \sin\omega\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,. \end{aligned} \tag{14}$$

From these follow by differentiation

$$\begin{aligned} \frac{\partial X_1'}{\partial u} &= -\frac{\partial\theta}{\partial u} X_2' - \sin\theta\, X', & \frac{\partial X_1'}{\partial v} &= \frac{\partial\theta}{\partial v} X_2' + \sin\theta\, X', \\ \frac{\partial X_2'}{\partial u} &= \frac{\partial\theta}{\partial v} X_1' - \cos\theta\, X', & \frac{\partial X_2'}{\partial v} &= -\frac{\partial\theta}{\partial v} X_1' - \cos\theta\, X', \end{aligned} \tag{15}$$

equations which are analogous to (3). Equations (14) are equivalent to

$$\begin{aligned} \cos\theta\, X_1' + \sin\theta\, X_2' &= -\sin(\theta-\omega)\,\big[\cos\sigma\,(\sin\theta\, X_1 \\ &\qquad - \cos\theta\, X_2) - \sin\sigma\, X\big] - \cos(\theta-\omega)\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,, \\ \cos\theta\, X_1' - \sin\theta\, X_2' &= \sin(\theta+\omega)\,\big[\cos\sigma\,(\sin\theta\, X_1 \\ &\qquad - \cos\theta\, X_2) - \sin\sigma\, X\big] - \cos(\theta+\omega)\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,. \end{aligned} \tag{16}$$

2. The Transformation $\Omega(\theta, \sigma)$ of the Surfaces V

Each solution W_1 of the equation

$$\frac{\partial^2\phi_1}{\partial u\,\partial v} + \cos 2\theta\,\phi_1 = 0 \tag{17}$$

* Equations (10) may be obtained from equations (44) E., p. 289, by replacing $\cos\sigma$ by $-\cos\sigma$.

† This follows from E., p. 284.

‡ E., l. c.

determines a surface V with the spherical representation (9) of its geodesic conjugate system. The rectangular coördinates of the corresponding surface V_1 are of the form

$$x_1 = W_1 X' + \frac{1}{\sin 2\theta}\left[X_1' \cos\theta \left(\frac{\partial W_1}{\partial u} - \frac{\partial W_1}{\partial v} \right) + X_2' \sin\theta \left(\frac{\partial W_1}{\partial u} + \frac{\partial W_1}{\partial v} \right) \right], \tag{18}$$

where X', X_1', and X_2' have the significance given by equations (11) and (14).

It is our purpose to show that when a surface V whose coördinates are given by (5) is known, it is possible to find a surface V_1 defined by (18), such that the developables of the congruence formed by the joins of corresponding points of V and V_1 cut these surfaces in their respective geodesic conjugate systems.

If we write the coördinates of any point on a line of this congruence in the form

$$x + t(x_1 - x), \qquad y + t(y_1 - y), \qquad z + t(z_1 - z). \tag{19}$$

it must be possible to find two values of t, say t_1 and t_2, such that

$$\begin{aligned} \frac{(1-t_1)\frac{\partial x}{\partial u} + t_1 \frac{\partial x_1}{\partial u}}{x_1 - x} &= \frac{(1-t_1)\frac{\partial y}{\partial u} + t_1 \frac{\partial y_1}{\partial u}}{y_1 - y} = \frac{(1-t_1)\frac{\partial z}{\partial u} + t_1 \frac{\partial z_1}{\partial u}}{z_1 - z}, \\ \frac{(1-t_2)\frac{\partial x}{\partial v} + t_2 \frac{\partial x_1}{\partial v}}{x_1 - x} &= \frac{(1-t_2)\frac{\partial y}{\partial v} + t_2 \frac{\partial y_1}{\partial v}}{y_1 - y} = \frac{(1-t_2)\frac{\partial z}{\partial v} + t_2 \frac{\partial z_1}{\partial v}}{z_1 - z}. \end{aligned} \tag{20}$$

The necessary and sufficient condition that these equations be consistent is that the following equations be satisfied:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x_1}{\partial u} & \frac{\partial y_1}{\partial u} & \frac{\partial z_1}{\partial u} \\ x_1 - x & y_1 - y & z_1 - z \end{vmatrix} = 0, \qquad \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x_1}{\partial v} & \frac{\partial y_1}{\partial v} & \frac{\partial z_1}{\partial v} \\ x_1 - x & y_1 - y & z_1 - z \end{vmatrix} = 0. \tag{21}$$

The equations for V_1 analogous to (6) are

$$\begin{aligned} \frac{\partial x_1}{\partial u} &= -\frac{D_1}{\sin 2\theta}(\cos\theta\, X_1' + \sin\theta\, X_2'), \\ \frac{\partial x_1}{\partial v} &= \frac{D_1''}{\sin 2\theta}(\cos\theta\, X_1' - \sin\theta\, X_2'). \end{aligned} \tag{22}$$

Again with the aid of equations (11) and (14) we obtain from (5) and (18) the expression

$$x_1 - x = AX + BX_1 + CX_2, \tag{23}$$

where

$$\begin{aligned}
A &= W_1 \cos\sigma - W + \frac{\sin\sigma}{\sin 2\theta}\left(\sin(\theta-\omega)\frac{\partial W_1}{\partial u} + \sin(\theta+\omega)\frac{\partial W_1}{\partial v}\right),\\
B &= W_1 \sin\sigma \sin\theta - \frac{\cos\sigma \sin\theta}{\sin 2\theta}\left(\sin(\theta-\omega)\frac{\partial W_1}{\partial u}\right.\\
&\qquad \left. + \sin(\theta+\omega)\frac{\partial W_1}{\partial v}\right) - \frac{\cos\omega}{\sin 2\omega}\left(\frac{\partial W}{\partial u} - \frac{\partial W}{\partial v}\right)\\
&\qquad + \frac{\cos\theta}{\sin 2\theta}\left(\cos(\theta+\omega)\frac{\partial W_1}{\partial v} - \cos(\theta-\omega)\frac{\partial W_1}{\partial u}\right),\\
C &= -W_1 \sin\sigma\cos\theta + \frac{\cos\sigma\cos\theta}{\sin 2\theta}\left(\sin(\theta-\omega)\frac{\partial W_1}{\partial u}\right.\\
&\qquad \left. + \sin(\theta+\omega)\frac{\partial W_1}{\partial v}\right) - \frac{\sin\omega}{\sin 2\omega}\left(\frac{\partial W}{\partial u} + \frac{\partial W}{\partial v}\right)\\
&\qquad + \frac{\sin\theta}{\sin 2\theta}\left(\cos(\theta+\omega)\frac{\partial W_1}{\partial v} - \cos(\theta-\omega)\frac{\partial W_1}{\partial u}\right).
\end{aligned} \tag{24}$$

When these values are substituted in (21) and use is made of (16) in the reduction, we obtain

$$\begin{aligned}
A\cos(\theta-\omega)(\cos\sigma - 1) + B\sin\omega\sin\sigma - C\cos\omega\sin\sigma = 0,\\
A\cos(\theta+\omega)(\cos\sigma + 1) - B\sin\omega\sin\sigma - C\cos\omega\sin\sigma = 0.
\end{aligned}$$

In consequence of (24) these equations, and consequently equations (21), are equivalent to

$$\begin{aligned}
\frac{\partial W_1}{\partial u} + \frac{\cos\sigma + 1}{\sin\sigma}\cos(\theta+\omega)W_1 &= -\frac{\partial W}{\partial u} + \frac{1+\cos\sigma}{\sin\sigma}\cos(\theta+\omega)W,\\
\frac{\partial W_1}{\partial v} + \frac{\cos\sigma - 1}{\sin\sigma}\cos(\theta-\omega)W_1 &= \frac{\partial W}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\cos(\theta-\omega)W.
\end{aligned} \tag{25}$$

Since θ satisfies equations (10), the condition of integrability of equations (25) is satisfied, as is readily shown, and furthermore the function W_1 so defined is a solution of equation (17). We shall find shortly (§ 5) that the integration of equations (25) requires one quadrature only. Hence we have

THEOREM I. *Each solution of equations* (10) *determines a transformation* $\Omega(\theta, \sigma)$ *of a given surface* V *into a surface* V_1 *such that the developables of the congruence formed by the joins of corresponding points on* V *and* V_1 *cut these*

surfaces in the geodesic conjugate systems. Moreover, when θ is known, the further determination of V_1 requires only a quadrature.

The second fundamental coefficients of V_1, namely D_1, D_1'', are given by expressions similar to (7). In consequence of (25) these can be given the form

$$D_1 = -D - \frac{2A}{\sin\sigma\sin 2\omega}\left(2\cos(\theta-\omega)\frac{\partial\omega}{\partial u} - \frac{\cos\sigma+1}{\sin\sigma}\sin 2\omega\right),$$

(26)

$$D_1'' = D'' - \frac{2A}{\sin\sigma\sin 2\omega}\left(2\cos(\theta+\omega)\frac{\partial\omega}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\sin 2\omega\right).$$

3. The Congruence Associated with a Transformation $\Omega(\theta, \sigma)$

With the aid of (25) the expressions (24) are reducible to

$$A = \frac{\sin\sigma}{\sin 2\theta}\left(\frac{\sin 2\omega}{\sin\sigma}(W_1 - W\cos\sigma) - \sin(\theta-\omega)\frac{\partial W}{\partial u} + \sin(\theta+\omega)\frac{\partial W}{\partial v}\right),$$

(27)

$$B = \frac{A}{\sin\sigma\sin\omega}(\cos\theta\cos\omega - \cos\sigma\sin\theta\sin\omega),$$

$$C = \frac{A}{\sin\sigma\cos\omega}(\cos\theta\cos\omega\cos\sigma - \sin\theta\sin\omega).$$

From these expressions and (23) it follows that the direction-cosines α, β, and γ of the line joining corresponding points on V and V_1 are of the form

$$\alpha\rho = \cos\omega(\cos\theta\cos\omega - \cos\sigma\sin\omega\sin\theta)X_1 + \sin\omega(\cos\theta\cos\omega\cos\sigma - \sin\theta\sin\omega)X_2 + \sin\sigma\sin\omega\cos\omega X,$$

(28)

where

$$\rho = \sqrt{2}\sqrt{1+\cos 2\omega\cos 2\theta - \cos\sigma\sin 2\omega\sin 2\theta}.$$

(28)

Since the expressions for these direction-cosines involve only the functions of the Bäcklund transformation, we have

Theorem II. *If V and V' are any two surfaces conjugate to P, and V_1 and V_1' are obtained by transformations involving the same functions θ and σ, the developables of the congruences of lines joining corresponding points of V and V_1 and of V' and V_1' have the same spherical representation.*

In § 4 we shall find that in certain cases these congruences coincide.

In the tangent plane to V at a point M we draw the line l parallel to the line joining corresponding points of the pseudospherical surfaces P and P_1 determining a transformation of V. The direction-cosines $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ of

this line are of the form

$$\bar{X} = \cos\theta\, X_1 + \sin\theta\, X_2. \tag{29}$$

The direction-cosines $\bar{X}'$, $\bar{Y}'$, and $\bar{Z}'$ of the line in this plane perpendicular to l are of the form

$$\bar{X}' = -\sin\theta\, X_1 + \cos\theta\, X_2. \tag{30}$$

With the aid of these expressions equation (28) may be written

$$\begin{aligned} 2\alpha\rho = {} & \sin\sigma\sin 2\omega\, X + (\cos 2\omega + \cos 2\theta)\,\bar{X} \\ & + (-\sin 2\theta + \cos\sigma\sin 2\omega)\,\bar{X}'. \end{aligned} \tag{31}$$

In view of this expression we are able to give the following construction for the direction of the line through M upon which lies the point M_1 of the transformed surface V_1. In the tangent plane to V at M we lay off a unit vector which makes the angle -2θ with the line l, and in the plane through l and inclined at the angle σ to the tangent plane we lay off a unit vector which makes the angle 2ω with l; the vector which is the sum of these two vectors has the desired direction.

It is our purpose now to determine the abscissas of the focal points of the congruence of joins of corresponding points on V and V_1. To this end we observe that in consequence of (20) we have

$$\frac{(1-t_1)\sum X'\dfrac{\partial x}{\partial u}}{\sum X'(x_1-x)} = \frac{t_1\sum X\dfrac{\partial x_1}{\partial u}}{\sum X(x_1-x)}, \qquad \frac{(1-t_2)\sum X'\dfrac{\partial x}{\partial v}}{\sum X'(x_1-x)} = \frac{t_2\sum X\dfrac{\partial x_1}{\partial v}}{\sum X(x_1-x)}.$$

It is readily shown that

$$\sum X\frac{\partial x_1}{\partial u} = -\frac{D_1}{\sin 2\theta}\sin\sigma\sin(\theta-\omega),$$

$$\sum X\frac{\partial x_1}{\partial v} = -\frac{D_1''}{\sin 2\theta}\sin\sigma\sin(\theta+\omega),$$

$$\sum X'\frac{\partial x}{\partial u} = -\frac{D}{\sin 2\omega}\sin\sigma\sin(\theta-\omega),$$

$$\sum X'\frac{\partial x}{\partial v} = \frac{D''}{\sin 2\omega}\sin\sigma\sin(\theta+\omega).$$

In consequence of the values (27) we have also

$$\sum X'(x_1-x) = \frac{\sin 2\theta}{\sin 2\omega}A.$$

Substituting these expressions in the above equations, we obtain

$$(1-t_1)D = t_1 D_1, \qquad (1-t_2)D'' = -t_2 D_1'',$$

from which we derive the following:

$$t_1 = \frac{D}{D + D_1}, \qquad 1 - t_1 = \frac{D_1}{D + D_1},$$
$$t_2 = \frac{D''}{D'' - D_1''}, \qquad 1 - t_2 = \frac{-D_1''}{D'' - D_1''}. \tag{32}$$

From these values it follows that the cross-ratio of the points M and M_1 on V and V_1 and the foci F_1 and F_2 of their joins is

$$R(MM_1, F_1F_2) = -\frac{DD_1''}{D''D_1}. \tag{33}$$

Substituting the values (32) in (19), we find that the coördinates of the focal points are of the form

$$\frac{xD_1 + x_1D}{D_1 + D}, \text{ etc.}; \qquad \frac{xD_1'' - x_1D''}{D_1'' - D''}, \text{ etc.} \tag{34}$$

Hence we have

THEOREM III. *The focal points of the lines joining corresponding points M and M_1 on the surfaces V and V_1 divide the segment MM_1 in the ratios D/D_1 and $-D''/D_1''$.*

4. SPECIAL SURFACES OF VOSS

In the preceding discussion we have ignored the possibility that the joins of corresponding points on V and V_1 may pass through a point, in which case any ruled surface of the congruence is developable and Theorem I loses part of its significance. It is our purpose now to consider this possibility. We shall find that every surface V does not admit of such a transformation. We use w to denote the function W for a surface V which possesses this property.

If we take the origin for the point of concurrence of all the rays, we must have

$$x + t(x_1 - x) = 0, \qquad y + t(y_1 - y) = 0, \qquad z + t(z_1 - z) = 0.$$

When the expressions from (5) and (23) are substituted in these equations, we obtain three conditions of the form

$$LX + MX_1 + NX_2 = 0,$$

which coexist only in case $L = M = N = 0$. This gives the three equations of condition

$$w + At = 0, \qquad \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} + 2\sin\omega tB = 0,$$
$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} + 2\cos\omega tC = 0. \tag{35}$$

If we solve the last two equations for $\partial w/\partial u$ and $\partial w/\partial v$ and substitute the values of B and C from (27), we have

$$\frac{\partial w}{\partial u} + tA\,\frac{(1+\cos\sigma)}{\sin\sigma}\cos(\theta+\omega) = 0, \qquad \frac{\partial w}{\partial v} + tA\,\frac{\cos\sigma - 1}{\sin\sigma}\cos(\theta-\omega).$$

Hence equations (35) may be replaced by the first of them and

$$(36)\qquad \frac{\partial \log w}{\partial u} = \frac{1+\cos\sigma}{\sin\sigma}\cos(\theta+\omega), \qquad \frac{\partial \log w}{\partial v} = \frac{\cos\sigma - 1}{\sin\sigma}\cos(\theta-\omega).$$

From the equations (25) for the determination of w_1 it follows that

$$(37)\qquad ww_1 = \text{const.}$$

Hence each solution θ of equations (10) leads by a quadrature to a surface V, which we call a *special surface of Voss* and denote by $V(\theta, \sigma)$, such that there is another surface of the same kind so related that the lines joining corresponding points of the two surfaces are concurrent. We observe that the relation between these surfaces is reciprocal, and consequently we say that each is the *adjoint* of the other.

It is evident that if there is applied to a special surface $V(\theta, \sigma)$ a transformation involving a different θ, the relation between the two surfaces will be that of Theorem I.

When the values (36) are substituted in equations (7), we find that the expressions for the second fundamental coefficients Δ, Δ'' of $V(\theta, \sigma)$ are reducible to

$$(38)\qquad \Delta = \frac{2w}{\sin\sigma}\left(\frac{2\cos(\theta-\omega)}{\sin 2\omega}\frac{\partial\omega}{\partial u} - \frac{1+\cos\sigma}{\sin\sigma}\right),$$
$$\Delta'' = -\frac{2w}{\sin\sigma}\left(\frac{2\cos(\theta+\omega)}{\sin 2\omega}\frac{\partial\omega}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\right).$$

From (26) it follows that the coefficients Δ_1, Δ_1'' of the adjoint surface are given by

$$(39)\qquad \Delta_1 = -\Delta\left(1+\frac{A}{w}\right), \qquad \Delta_1'' = \Delta''\left(1+\frac{A}{w}\right).$$

Since we have

$$(40)\qquad \Delta_1\Delta'' + \Delta_1''\Delta = 0,$$

it follows that the asymptotic lines on a special surface of Voss correspond to a conjugate system on its adjoint.

We consider the converse problem, that is, we suppose that D, D'', D_1, D_1'' satisfy an equation of the form (40). From (26) it follows that either $A = 0$,

or there exists a function λ such that

$$D = \frac{\lambda}{\sin 2\omega}\left(2\cos(\theta-\omega)\frac{\partial\omega}{\partial u} - \frac{1+\cos\sigma}{\sin\sigma}\sin 2\omega\right),$$
$$D'' = -\frac{\lambda}{\sin 2\omega}\left(2\cos(\theta+\omega)\frac{\partial\omega}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\sin 2\omega\right). \tag{41}$$

If we take $A = 0$ and substitute in (25) the value of W_1 so determined, we find that $D = D'' = 0$, which evidently is impossible.

When the values (41) are substituted in the Codazzi equations (8), we find

$$\lambda = cw.$$

Hence V is a special surface of Voss, and we have

THEOREM IV. *A necessary and sufficient condition that a surface of Voss be "special" is that it admit a transformation Ω such that the asymptotic lines of the given surface V correspond to a conjugate system on the surface V_1.*

5. EQUATIONS OF A TRANSFORMATION $\Omega(\theta, \sigma)$ IN ANOTHER FORM

In consequence of equations (36) we may put the fundamental equations (25) in the form

$$\frac{\partial}{\partial u}(W_1 w) = -w\frac{\partial W}{\partial u} + \frac{\partial w}{\partial u}W,$$
$$\frac{\partial}{\partial v}(W_1 w) = w\frac{\partial W}{\partial v} - \frac{\partial w}{\partial v}W. \tag{42}$$

This shows that the transformation $\Omega(\theta, \sigma)$ is of the general class of transformations of Moutard of equations with equal invariants.* It is a well-known fact that three linearly independent solutions ν_1, ν_2, ν_3 of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = M\theta \tag{43}$$

determine a surface Σ referred to its asymptotic lines; its coördinates ξ, η, ζ are given by the formulas of Lelieuvre†

$$\frac{\partial\xi}{\partial u} = \begin{vmatrix} \nu_2 & \nu_3 \\ \frac{\partial\nu_2}{\partial u} & \frac{\partial\nu_3}{\partial v} \end{vmatrix}, \qquad \frac{\partial\xi}{\partial v} = -\begin{vmatrix} \nu_2 & \nu_3 \\ \frac{\partial\nu_2}{\partial v} & \frac{\partial\nu_3}{\partial v} \end{vmatrix}. \tag{44}$$

If a fourth solution of equation (43) is known, say θ_1, the functions $\overline{\nu}_i$, defined

* Bianchi, *Lezioni di geometria differenziale*, vol. 2, p. 47; also Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, p. 145.

† E., p. 193.

by the quadratures

$$(45)\qquad \frac{\partial}{\partial u}(\theta_1\bar{\nu}_i) = -\begin{vmatrix} \theta_1 & \nu_i \\ \dfrac{\partial\theta_1}{\partial u} & \dfrac{\partial\nu_i}{\partial u}\end{vmatrix}, \qquad \frac{\partial}{\partial v}(\theta_1\bar{\nu}_i) = \begin{vmatrix} \theta_1 & \nu_i \\ \dfrac{\partial\theta_1}{\partial v} & \dfrac{\partial\nu_i}{\partial v}\end{vmatrix} \quad (i = 1, 2, 3)$$

satisfy the differential equation

$$(43')\qquad \frac{\partial^2\theta}{\partial u\partial v} = \theta_1\frac{\partial^2}{\partial u\partial v}\frac{1}{\theta_1}\cdot\theta.$$

Moreover, the surface Σ_1, determined by equations in $\bar{\nu}_i$ analogous to (44), can be so placed in space that Σ and Σ_1 are the focal surfaces of a W-congruence.*

Since the total curvature of Σ is given by †

$$K = \frac{-1}{(\nu_1^2+\nu_2^2+\nu_3^2)^2},$$

if Σ and Σ_1 are to be pseudospherical surfaces of curvature $-(1/a^2)$ in the relation of a transformation of Bäcklund,

$$\nu_1 = \sqrt{a}X, \qquad \nu_2 = \sqrt{a}Y, \qquad \nu_3 = \sqrt{a}Z,$$

$$\bar{\nu}_1 = \sqrt{a}X', \qquad \bar{\nu}_2 = \sqrt{a}Y', \qquad \bar{\nu}_3 = \sqrt{a}Z',$$

where X', Y', Z' are given by (11). When these values are substituted in (45), we find that θ_1 equals w, given by (36), to within a constant factor. Now, equations (43) and (43′) are respectively (4) and (17). Hence we have

THEOREM V. *A transformation $\Omega(\theta, \sigma)$ is a transformation of Moutard in which the transforming function is the tangential coördinate w of the corresponding special surface $V(\theta, \sigma)$.*

It is a fundamental property of W-congruences that each focal surface admits an infinitesimal deformation parallel to the normal to the other surface. The surface corresponding with orthogonality of linear elements to Σ, defined by (44), in this case has for coördinates ‡

$$\theta_1\bar{\nu}_1, \qquad \theta_1\bar{\nu}_2, \qquad \theta_1\nu_3.$$

Moreover, in the general case $\theta_1\sqrt[4]{-K}$ is the tangential coördinate W of the corresponding associate surface. Hence we have

THEOREM VI. *A special surface $V(\theta, \sigma)$ associated with a pseudospherical surface P is that associate surface in the infinitesimal deformation of P whose directrices are parallel to the corresponding normals to the Bäcklund transform of P by means of θ and σ.*

* E., pp. 418, 419.
† E., p. 194.
‡ E., p. 420.

Since $W_1 w$ is determined by (42) only to within an additive constant, and since also $1/w$ is a solution of equation (17), if V_1 is a transform of V by a transformation $\Omega(\theta, \sigma)$, so also is the suite of surfaces whose tangential coördinates are

$$X', \quad Y', \quad Z', \quad W_1 + \frac{a}{w},$$

where a is an arbitrary constant.

In like manner it may be observed that V_1 is obtained by a transformation $\Omega(\theta, \sigma)$ from any one of the surfaces whose tangential coördinates are

$$X, \quad Y, \quad Z, \quad W + aw,$$

provided it is true when $a = 0$.

We have seen that when a surface $V(\theta, \sigma)$ is transformed by $\Omega(\theta, \sigma)$, the tangential coördinates w and w_1 are such that their product is constant. We shall show that this property is characteristic of special surfaces. In fact, if we differentiate the equation

$$WW_1 = \text{const.}$$

with respect to u and v respectively and make use of equations (42), we get

$$(W_1 - W)\frac{\partial}{\partial u}\log\frac{W}{w} = 0, \qquad (W_1 - W)\frac{\partial}{\partial v}\log\frac{W}{w} = 0.$$

Hence V is a special surface $V(\theta, \sigma)$ and the transformation is $\Omega(\theta, \sigma)$. We shall say that in this case the two special surfaces are in perspective correspondence.

6. Theorem of Permutability of the Transformations Ω

It is our purpose to show that the transformations which have been established in §2 admit a "theorem of permutability" as follows:

Theorem VII. *If two surfaces V_1 and V_2 are obtained from a surface V by means of transformations $\Omega(\theta_1, \sigma_1)$ and $\Omega(\theta_2, \sigma_2)$ respectively, there exists a surface V' which may be obtained from V_1 by a transformation $\Omega(\phi, \sigma_2)$ and from V_2 by a transformation $\Omega(\phi, \sigma_1)$; moreover, the determination of V' does not involve integration.*

If W_1 and W_2 are tangential coördinates of V_1 and V_2 respectively, we must have in accordance with (42)

$$\begin{aligned} \frac{\partial W_i}{\partial u} + \frac{W_i}{w_i}\frac{\partial w_i}{\partial u} &= -\frac{\partial W}{\partial u} + \frac{W}{w_i}\frac{\partial w_i}{\partial u}, \\ \frac{\partial W_i}{\partial v} + \frac{W_i}{w_i}\frac{\partial w_i}{\partial v} &= \frac{\partial W}{\partial v} - \frac{W}{w_i}\frac{\partial w_i}{\partial v} \end{aligned} \qquad (i = 1, 2), \tag{46}$$

where the functions w_1 and w_2 are given by

$$\begin{aligned} \frac{\partial \log w_i}{\partial u} &= \frac{1+\cos\sigma_i}{\sin\sigma_i}\cos(\theta_i+\omega), \\ \frac{\partial \log w_i}{\partial v} &= \frac{\cos\sigma_i - 1}{\sin\sigma_i}\cos(\theta_i-\omega) \end{aligned} \qquad (i=1,2), \tag{47}$$

and

$$\begin{aligned} \sin\sigma_i\left(\frac{\partial\theta_i}{\partial u}-\frac{\partial\omega}{\partial u}\right)+(\cos\sigma_i+1)\sin(\theta_i+\omega)=0, \\ \sin\sigma_i\left(\frac{\partial\theta_i}{\partial v}+\frac{\partial\omega}{\partial v}\right)+(\cos\sigma_i-1)\sin(\theta_i-\omega)=0 \end{aligned} \qquad (i=1,2). \tag{48}$$

On the assumption that a surface V' exists satisfying the requirements of the above theorem, there must exist functions W', w_1', and w_2' satisfying equations analogous to (46), namely

$$\begin{aligned} \frac{\partial W'}{\partial u}+\frac{W'}{w_i'}\frac{\partial w_i'}{\partial u} &= -\frac{\partial W_i}{\partial u}+\frac{W_i}{w_i'}\frac{\partial w_i'}{\partial u}, \\ \frac{\partial W'}{\partial v}+\frac{W'}{w_i'}\frac{\partial w_i'}{\partial v} &= \frac{\partial W_i}{\partial v}-\frac{W_i}{w_i'}\frac{\partial w_i'}{\partial v} \end{aligned} \qquad (i=1,2). \tag{49}$$

By means of (46) these are reducible to

$$\begin{aligned} \frac{\partial W'}{\partial u}+W'\frac{\partial \log w_i'}{\partial u} &= W_i\cdot\frac{\partial}{\partial u}\log(w_i' w_i)+\frac{\partial W}{\partial u}-W\frac{\partial \log w_i}{\partial u}, \\ \frac{\partial W'}{\partial v}+W'\frac{\partial \log w_i'}{\partial v} &= -W_i\cdot\frac{\partial}{\partial v}\log(w_i' w_i)+\frac{\partial W}{\partial v}-W\frac{\partial \log w_i}{\partial v} \end{aligned} \qquad (i=1,2). \tag{50}$$

The consistency of the first equations for $i=1$ and $i=2$, and likewise for the second equations, requires that

$$\begin{aligned} W'\frac{\partial}{\partial u}\log\frac{w_1'}{w_2'}-W_1\frac{\partial}{\partial u}\log(w_1' w_1) \\ +W_2\frac{\partial}{\partial u}\log(w_2' w_2)+W\frac{\partial}{\partial u}\log\frac{w_1}{w_2}=0, \\ W'\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'}+W_1\frac{\partial}{\partial v}\log(w_1' w_1) \\ -W_2\frac{\partial}{\partial v}\log(w_2' w_2)+W\frac{\partial}{\partial v}\log\frac{w_1}{w_2}=0. \end{aligned} \tag{51}$$

Since V_1 and V_2 are general transforms of V, the coexistence of equations

(51) necessitates the conditions

$$\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial u}\log w_1' w_1 + \frac{\partial}{\partial u}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial v}\log w_1' w_1 = 0,$$

$$\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial u}\log w_2' w_2 + \frac{\partial}{\partial u}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial v}\log w_2' w_2 = 0, \tag{52}$$

$$\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial u}\log\frac{w_2}{w_1} - \frac{\partial}{\partial u}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial v}\log\frac{w_2}{w_1} = 0.$$

This system may be replaced by the first and the following equations

$$\frac{\partial}{\partial u}(w_1' w_1) = -\mu\left(w_1\frac{\partial w_2}{\partial u} - w_2\frac{\partial w_1}{\partial u}\right),$$

$$\frac{\partial}{\partial u}(w_2' w_2) = +\lambda\left(w_1\frac{\partial w_2}{\partial u} - w_2\frac{\partial w_1}{\partial u}\right),$$

(53)

$$\frac{\partial}{\partial v}(w_1' w_1) = \mu\left(w_1\frac{\partial w_2}{\partial v} - w_2\frac{\partial w_1}{\partial v}\right),$$

$$\frac{\partial}{\partial v}(w_2' w_2) = -\lambda\left(w_1\frac{\partial w_2}{\partial v} - w_2\frac{\partial w_1}{\partial v}\right),$$

where λ and μ are functions to be determined. When these values are substituted in the first of equations (52) we find that it may be replaced by

$$\mu w_2 w_2' + \lambda w_1 w_1' = 0. \tag{54}$$

In consequence of (53) and (54) equations (51) reduce to

$$W' = (W_1 - W_2)\lambda\frac{w_1}{w_2'} + W = (W_2 - W_1)\mu\frac{w_2}{w_1'} + W. \tag{55}$$

If we substitute these expressions in (50), we find that λ and μ must be constant.

It is necessary now that we determine whether functions w_2' and w_1' exist which satisfy equations (53) and also equations analogous to (47), namely

$$\frac{\partial\log w_i'}{\partial u} = \frac{1+\cos\sigma_k}{\sin\sigma_k}\cos(\phi+\theta_i),$$

$$\frac{\partial\log w_i'}{\partial v} = \frac{\cos\sigma_k - 1}{\sin\sigma_k}\cos(\phi-\theta_i) \qquad (i, k = 1, 2;\ i \neq k), \tag{56}$$

where the function ϕ must satisfy the equations

$$\sin\sigma_i\left(\frac{\partial\phi}{\partial u} - \frac{\partial\theta_k}{\partial u}\right) + (\cos\sigma_i + 1)\sin(\phi+\theta_k) = 0,$$

$$\sin\sigma_i\left(\frac{\partial\phi}{\partial v} + \frac{\partial\theta_k}{\partial v}\right) + (\cos\sigma_i - 1)\sin(\phi-\theta_k) = 0 \tag{57}$$

$$(i, k = 1, 2;\ i \neq k).$$

From the "theorem of permutability" of transformations of Bäcklund of pseudospherical surfaces it follows that there exists a function ϕ satisfying the four equations (57); it is given by *

$$\begin{aligned}\sin(\phi-\omega) &= \frac{(\cos\sigma_1-\cos\sigma_2)\sin(\theta_2-\theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1},\\ \cos(\phi-\omega) &= \frac{\sin\sigma_1\sin\sigma_2+(\cos\sigma_1\cos\sigma_2-1)\cos(\theta_2-\theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1}.\end{aligned} \tag{58}$$

Since w_1' and w_2' are given by (56) only to within a constant factor, it follows from (53) that the constants μ and λ may be taken equal to unity, and consequently equations (53) may be written

$$\begin{aligned}\frac{\partial}{\partial u}(w_i' w_i) &= w_k\frac{\partial w_i}{\partial u}-w_i\frac{\partial w_k}{\partial u},\\ \frac{\partial}{\partial v}(w_i' w_i) &= -w_k\frac{\partial w_i}{\partial v}+w_i\frac{\partial w_k}{\partial v}\end{aligned}\qquad (i,k=1,2;\ i\neq k). \tag{59}$$

If we reduce these equations for $i=1$, $k=2$ by means of (56) and (47), we obtain

$$\begin{aligned}&w_1'\left(\frac{1+\cos\sigma_2}{\sin\sigma_2}\cos(\phi+\theta_1)+\frac{1+\cos\sigma_1}{\sin\sigma_1}\cos(\theta_1+\omega)\right)\\ &\qquad=-w_2\left(\frac{1+\cos\sigma_2}{\sin\sigma_2}\cos(\theta_2+\omega)-\frac{1+\cos\sigma_1}{\sin\sigma_1}\cos(\theta_1+\omega)\right),\\ &w_1'\left(\frac{\cos\sigma_2-1}{\sin\sigma_2}\cos(\phi-\theta_1)+\frac{\cos\sigma_1-1}{\sin\sigma_1}\cos(\theta_1-\omega)\right)\\ &\qquad=w_2\left(\frac{\cos\sigma_2-1}{\sin\sigma_2}\cos(\theta_2-\omega)-\frac{\cos\sigma_1-1}{\sin\sigma_1}\cos(\theta_1-\omega)\right).\end{aligned} \tag{60}$$

If we add these equations, the result is reducible to

$$\begin{aligned}&\frac{w_1'}{w_2}\Big[(\cot\sigma_2\cos\phi+\cot\sigma_1\cos\omega)\cos\theta_1\\ &\qquad-\left(\frac{1}{\sin\sigma_2}\sin\phi+\frac{1}{\sin\sigma_1}\sin\omega\right)\sin\theta_1\Big]\\ &=\left(-\frac{\cos\theta_2}{\sin\sigma_2}+\frac{\cos\theta_1}{\sin\sigma_1}\right)\cos\omega+(\cot\sigma_2\sin\theta_2-\cot\sigma_1\sin\theta_1)\sin\omega.\end{aligned} \tag{61}$$

By means of the values of ϕ given by (58) this equation becomes

$$\frac{w_1'}{w_2}=\frac{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1}{\cos\sigma_2-\cos\sigma_1}. \tag{62}$$

* E., p. 287.

The same result is obtained if equations (60) be subtracted from each other and in the resulting equation the values of the functions of ϕ given by (58) be substituted. It is readily shown with the aid of (58) that this expression for w_1' satisfies equations (56) with $i = 1, k = 2$.

If we proceed in a similar manner with equations (59) with $i = 2, k = 1$, we obtain

$$\frac{w_2'}{w_1} = -\frac{\sin\sigma_1 \sin\sigma_2 \cos(\theta_2 - \theta_1) + \cos\sigma_1 \cos\sigma_2 - 1}{\cos\sigma_2 - \cos\sigma_1}. \tag{63}$$

These values of w' and w_2' are in agreement with (54) which now is

$$w_2' w_2 + w_1' w_1 = 0. \tag{64}$$

From (55) it follows that the tangential coördinate W' of the surface V is

$$W' = W + \frac{(W_2 - W_1)(\cos\sigma_2 - \cos\sigma_1)}{\sin\sigma_1 \sin\sigma_2 \cos(\theta_2 - \theta_1) + \cos\sigma_1 \cos\sigma_2 - 1}. \tag{65}$$

Hence the "theorem of permutability," as stated at the beginning of this section, has been completely established.

Suppose that $W = w_1$ and $W_1 = 1/w_1$; it is evident that the character of W_2 and W' depend upon the choice of the function w_2. If we require that V_2 and V' shall be special surfaces of Voss such that the joins of corresponding points are concurrent, in all generality we can put $W' W_2 = 1$. It follows at once from (65) and (63) that $W_2 = w_2'$. Now V_1 and V' are not in perspective correspondence. Otherwise we should have $w_1' w_1 = \text{const.}$, which is inconsistent with (59).

If four surfaces are related in accordance with Theorem VII, we say that they form a *quatern*. In view of the above results we have

THEOREM VIII. *If V, V_1, V_2, V' form a quatern of surfaces of Voss determined by a set of functions w_1, w_2, w_1', w_2', the special surfaces of Voss determined by the functions $w_1, 1/w_1, w_2', 1/w_2'$ form a quatern, and so likewise do the special surfaces determined by the functions $w_2, 1/w_2, w_1', 1/w_1'$.*

Conversely,

THEOREM IX. *If two surfaces V and V_1 of a quatern (V, V_1, V_2, V') are special surfaces in perspective correspondence, the necessary and sufficient condition that V_2 and V' be special surfaces in perspective correspondence is that these surfaces be determined by w_2' and $1/w_2'$, where w_2' is given by* (63), *σ_2 being any constant and θ_2 a solution of the corresponding equations* (10).

7. NORMAL CONGRUENCES DETERMINED BY TRANSFORMATIONS Ω

We turn now to the consideration of the congruence of the lines of intersection of the tangent planes at the corresponding points M and M_1 on two

surfaces V and V_1, of which the latter arises from V by a transformation Ω. From the general theory of congruences* we know that the tangents at M and M_1 to the geodesics $v = \text{const.}$ through these respective points meet in a point F_1; likewise, the tangents at M and M_1 to the geodesics $u = \text{const.}$ meet in a point F_2; furthermore, these points F_1 and F_2 are the focal points of their join in the congruence formed by these lines of intersection of the tangent planes to V and V_1 at M and M_1.

If the coördinates of F_1 and F_2 be denoted by $a_1, b_1, c_1; a_2, b_2, c_2$ respectively,

$$(66)\qquad \begin{aligned} a_1 &= x + \lambda_1(\cos\omega\, X_1 + \sin\omega\, X_2) = x_1 + \mu_1(\cos\theta\, X_1' + \sin\theta\, X_2'),\\ a_2 &= x + \lambda_2(\cos\omega\, X_1 - \sin\omega\, X_2) = x_1 + \mu_2(\cos\theta\, X_1' - \sin\theta\, X_2'), \end{aligned}$$

where $\lambda_1, \mu_1, \lambda_2, \mu_2$ are quantities to be determined. From (16) and (23) it follows that the equivalence of the two expressions for a_1, b_1, and c_1 requires that λ_1 and μ_1 satisfy

$$A + \mu_1 \sin\sigma \sin(\theta - \omega) = 0,$$

$$B - \mu_1[\sin(\theta - \omega)\cos\sigma \sin\theta + \cos(\theta - \omega)\cos\theta] - \lambda_1 \cos\omega = 0,$$

$$C + \mu_1[\sin(\theta - \omega)\cos\sigma \cos\theta - \cos(\theta - \omega)\sin\theta] - \lambda_1 \sin\omega = 0.$$

It is found that these equations are consistent in view of the values (27) of A, B, and C.

From the above equations we find

$$(67)\qquad \lambda_1 = \frac{A \sin 2\theta}{\sin\sigma \sin 2\omega \sin(\theta - \omega)}, \qquad \mu_1 = \frac{-A}{\sin\sigma \sin(\theta - \omega)}.$$

In like manner it can be shown that

$$(68)\qquad \lambda_2 = \frac{A \sin 2\theta}{\sin\sigma \sin 2\omega \sin(\theta + \omega)}, \qquad \mu_2 = \frac{A}{\sin\sigma \sin(\theta + \omega)}.$$

We shall need the expressions for $\partial A/\partial u$ and $\partial A/\partial v$. They are most easily obtained indirectly by differentiating equation (23) with respect to u and v separately and equating to zero the coefficients of X in each case. These expressions are reducible to

$$(69)\qquad \begin{aligned} \frac{\partial A}{\partial u} &= -\frac{D_1}{\sin 2\theta}\sin\sigma \sin(\theta - \omega) + \frac{A(1 + \cos\sigma)\cos(\theta + \omega)}{\sin\sigma},\\ \frac{\partial A}{\partial v} &= \frac{-D_1'}{\sin 2\theta}\sin\sigma \sin(\theta + \omega) + \frac{A(\cos\sigma - 1)\cos(\theta - \omega)}{\sin\sigma}. \end{aligned}$$

If we substitute in (66) the values of λ_1 and λ_2 given above, we derive by

* Cf. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, pp. 230, 231.

means of (69) the following:

$$\frac{\partial a_1}{\partial u} = \frac{\lambda_1 \sin 2\omega}{\sin(\theta-\omega)}\left(\frac{\cos\sigma+1}{\sin\sigma}(\cos\omega X_1+\sin\omega X_2)-\sin(\theta-\omega)X\right),$$

$$\frac{\partial a_1}{\partial v} = \left(2\lambda_1\frac{\partial\omega}{\partial v}-D''\right)\frac{(\cos\theta X_1+\sin\theta X_2)}{\sin(\theta-\omega)}, \tag{70}$$

$$\frac{\partial a_2}{\partial u} = -\left(2\lambda_2\frac{\partial\omega}{\partial u}+D\right)\frac{(\cos\theta X_1+\sin\theta X_2)}{\sin(\theta+\omega)},$$

$$\frac{\partial a_2}{\partial v} = \frac{\lambda_2\sin 2\omega}{\sin(\theta+\omega)}\left(\frac{1-\cos\sigma}{\sin\sigma}(\cos\omega X_1-\sin\omega X_2)+\sin(\theta+\omega)X\right). \tag{71}$$

These equations show incidentally that F_1 and F_2 are the focal points of the congruence.

We denote by Σ_1 and Σ_2 the focal surfaces of the congruence, that is the loci of the points F_1 and F_2, and by A_1, B_1, C_1, A_2, B_2, and C_2 the direction-cosines of the normals to Σ_1 and Σ_2. From (70) and (71) we find

$$A_1 = \left(\sin\theta X_1-\cos\theta X_2+\frac{1+\cos\sigma}{\sin\sigma}X\right)\frac{\sin\sigma}{\sqrt{2(1+\cos\sigma)}},$$

$$A_2 = \left(\sin\theta X_1-\cos\theta X_2+\frac{\cos\sigma-1}{\sin\sigma}X\right)\frac{\sin\sigma}{\sqrt{2(1-\cos\sigma)}}, \tag{72}$$

and similar expressions for the B's and C's.

Hence we have

$$\sum XA_1 = \cos\frac{\sigma}{2}, \qquad \sum X'A_1 = \cos\frac{\sigma}{2},$$

$$\sum XA_2 = -\sin\frac{\sigma}{2}, \qquad \sum X'A_2 = \sin\frac{\sigma}{2}. \tag{73}$$

Consequently we have

THEOREM X. *For the congruence of the lines of intersection of the tangent planes at corresponding points on V_1 and V the focal planes bisect the angles between the tangent planes, and consequently the congruence is normal.*

From (66) it follows that

$$a_1-a_2 = \frac{A\sin 2\theta}{\sin\sigma\sin(\theta-\omega)\sin(\theta+\omega)}(X_1\cos\theta+X_2\sin\theta), \tag{74}$$

consequently the focal distance 2ρ is given by

$$2\rho = \frac{A\sin 2\theta}{\sin\sigma\sin(\theta-\omega)\sin(\theta+\omega)}. \tag{75}$$

7. Converse Problem

Suppose that we have a pseudospherical congruence, that is the congruence of lines joining points of a pseudospherical surface P to the corresponding points of a Bäcklund transform P_1. Using the notation of the preceding sections, we note that the direction-cosines of this congruence, namely $\bar{X}$, $\bar{Y}$, and $\bar{Z}$, are of the form

$$\bar{X} = \cos\theta\, X_1 + \sin\theta\, X_2. \tag{76}$$

By differentiation we obtain

$$\begin{aligned} \frac{\partial \bar{X}}{\partial u} &= \sin(\theta+\omega)\left[(\sin\theta\, X_1 - \cos\theta\, X_2)\frac{\cos\sigma+1}{\sin\sigma} - X\right], \\ \frac{\partial \bar{X}}{\partial v} &= \sin(\theta-\omega)\left[(\sin\theta\, X_1 - \cos\theta\, X_2)\frac{\cos\sigma-1}{\sin\sigma} - X\right]. \end{aligned} \tag{77}$$

From these expressions we derive the following expressions for the coefficients of the linear element of the spherical representation of this congruence:

$$\sqrt{\mathcal{E}} = \frac{\sin(\theta+\omega)}{\sin\frac{1}{2}\sigma}, \qquad \mathcal{F} = 0, \qquad \sqrt{\mathcal{G}} = \frac{\sin(\theta-\omega)}{\cos\frac{1}{2}\sigma}. \tag{78}$$

It must be remembered that the parametric curves on the unit sphere correspond to the asymptotic lines on the focal surfaces of the congruence. Since this system on the sphere is orthogonal, it represents the lines of curvature of a group of surfaces $\bar{S}$ whose fundamental functions $\bar{D}$ and $\bar{D}''$ satisfy the Codazzi equations

$$\frac{\partial}{\partial v}\left(\frac{\bar{D}}{\sqrt{\mathcal{E}}}\right) = \frac{\bar{D}''}{\mathcal{G}}\frac{\partial\sqrt{\mathcal{E}}}{\partial v}, \qquad \frac{\partial}{\partial u}\left(\frac{\bar{D}''}{\sqrt{\mathcal{G}}}\right) = \frac{\bar{D}}{\mathcal{E}}\frac{\partial\sqrt{\mathcal{G}}}{\partial u}. \tag{79}$$

Each set of functions $\bar{D}$, $\bar{D}''$ satisfying these equations gives a surface $\bar{S}$ by means of the quadratures *

$$\frac{\partial \bar{x}}{\partial u} = -\frac{\bar{D}}{\mathcal{E}}\frac{\partial \bar{X}}{\partial u}, \qquad \frac{\partial \bar{x}}{\partial v} = -\frac{\bar{D}''}{\mathcal{G}}\frac{\partial \bar{X}}{\partial v}. \tag{80}$$

Suppose we have such a surface $\bar{S}$. Through each normal we draw planes parallel to the corresponding tangent planes to the pseudospherical surfaces P and P_1. The tangential coördinates W and W_1 of the envelopes of these planes are given by

$$W = \sum \bar{x}X, \qquad W_1 = \sum \bar{x}X'.$$

* E., p. 161.

With the aid of (79) and (80) we show that

$$\frac{\partial^2 W}{\partial u \partial v} + \cos 2\omega\, W = 0, \qquad \frac{\partial^2 W_1}{\partial u \partial v} + \cos 2\theta\, W_1 = 0.$$

Hence the envelopes of these planes are surfaces of Voss, and we have

THEOREM XI. *If P and P_1 are any two pseudospherical surfaces in the relation of a Bäcklund transformation, there exists a family of surfaces $\bar{S}$ each of which has its normals parallel to lines joining corresponding points of P and P_1, and, in this correspondence between the surfaces $\bar{S}$, P, and P_1, the lines of curvature on the first correspond to the asymptotic lines on the last two surfaces. Moreover, the two planes through each normal to $\bar{S}$ and parallel to the tangent planes to P and P_1 at the corresponding points envelope two surfaces of Voss which are in the relation of a transformation Ω.*

PRINCETON,
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BIRATIONAL TRANSFORMATIONS OF CERTAIN QUARTIC SURFACES*

BY

F. R. SHARPE AND VIRGIL SNYDER

1. **Statement of the problem.** It has been shown by Enriques† that if an algebraic surface contains a pencil of elliptic curves and also two curves cutting each curve of the pencil in two groups of points which are not equivalent and which do not possess equivalent multiples, the surface is invariant under a discontinuous infinite series of birational transformations. In a more recent memoir‡ the same author states that, with the exception of the case in which

$$p_g = p_a = P_2 = p^{(1)} = 1,$$

the converse is also true, since transversal curves of each curve of the pencil can always be found. Since the publication of the second theorem the author has discovered that the conclusion must be modified.§ An interesting example of a quartic surface having an infinite discontinuous group of birational transformations but not containing any elliptic curves has been found by Fano.‖ In the following paper it is proposed to study the possibility and the nature of birational transformations of quartic surfaces which pass through one or more given curves. The method of the basis of a system of curves on the surface will be employed, as developed by Severi.¶

* Presented to the Society, February 28, 1914.

† F. Enriques, *Sulle superficie algebriche, che ammettono una serie discontinua di trasformazioni birazionali*, Rendiconti della Reale Accademia dei Lincei, ser. 5, vol. 15 (1906), pp. 665–669.

‡ F. Enriques, *Sulle superficie algebriche con un fascio di curve ellittiche*, Rendiconti della Reale Accademia dei Lincei, ser. 5, vol. 21 (1912), pp. 14–17.

§ A. Rosenblatt, *Sur certaines classes des surfaces algébriques irregulières et sur les transformations birationnelles de ces surfaces en ellesmêmes*, Bulletin de l'académie des sciences de Cracovie, Juillet, 1912, pp. 761–810. See footnote 1, page 802.

‖ G. Fano, *Sopra alcune superficie del 4. ordine rappresentabili sul piano doppio*, Rendiconti del Reale Istituto Lombardo, ser. 2, vol. 39 (1906), pp. 1071–1086.

¶ F. Severi, *Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica*, Mathematische Annalen, vol. 62 (1906), pp. 194–225.

1. Quartic Surfaces Containing a Straight Line

2. **General case.** If a quartic surface contains a straight line it contains a pencil of plane elliptic cubics. If it is otherwise unrestricted, the surface contains no other rational curves, and consequently no birational transformations into itself.

3. **Rational curve on the surface.** Let the quartic surface contain the straight line C_1 and a rational curve C_n of order n meeting C_1 in $n-1$ points. Any plane through C_1 contains a plane cubic curve C_3 on F_4 that meets C_n in one point K. The involution defined by projecting C_3 upon itself from K leaves C_3 invariant. Associated with this involution is a Cremona transformation of the Jonquières type, having the polar conic of K with respect to C_3 for invariant curve. This Jonquières transformation will be understood when we speak of the involution. As K describes C_n the involution defines a Cremona transformation which leaves F_4 invariant and transforms every C_3 of the pencil into itself. When the equations of the surface and of C_1 and C_n are known, the equations of the transformation can be readily derived.

The system of curves C_1, $|C_3|$, and C_n constitute a basis, since we are concerned with the most general surface which satisfies the conditions.* The values of $[C_i, C_k]$ are

$$[C_1, C_1] = -2, \quad [C_3, C_3] = 0, \quad [C_n, C_n] = -2,$$
$$[C_1, C_3] = 3, \quad [C_1, C_n] = n-1, \quad [C_3, C_n] = 1.$$

In terms of the above curves the transformation may be expressed symbolically as follows.

Let C_1, C_3, C_n go into $\overline{C}_1$, $\overline{C}_3$, $\overline{C}_n$ respectively. The equations connecting the old and the new curves are found to be

$$T. \qquad \begin{aligned} \overline{C}_1 &= 2C_1 + 2(n+1)C_3 - 3C_n, \\ \overline{C}_3 &= C_3, \\ \overline{C}_n &= C_1 + 2(n+1)C_3 - 2C_n, \end{aligned}$$

by the following considerations. To obtain $\overline{C}_1$ we determine the image of C_1 as K describes C_n. Draw the lines from K to the three points of intersection of C_1 and the plane cubic curve defined by C_1 and K. As K describes C_n these joining lines generate a ruled surface R containing $\overline{C}_1$. On this ruled surface C_n is a threefold curve, and C_1 is an $(r-3)$-fold straight line, r being the order of R. The intersection of R with F_4 is a curve of order $4r$, consisting of C_n counted three times, of C_1 counted $r-2$ times, since the surfaces touch each other at every point of C_1, and of a residual curve, the

* Severi, l. c.

image of C_1. The curve C_n meets C_1 in $n-1$ points, corresponding to each one of which C_1 counts as a double generator of R, hence $r-4=2n-2$. The residual curve therefore has the symbol

$$2(n+1)(C_3+C_1)-2nC_1-3C_n$$

and is of order $2(n+1)4-2n-3n$.

It is evident that each plane cubic curve C_3 goes into itself.

The image of C_n is the locus of the first tangential of K as K describes C_n. The tangents to $|C_3|$ at the points of C_n describe a ruled surface of order $2(n+1)$ passing through C_n and touching F_4 at each point of C_n, and having C_1 as a $(2n+1)$-fold line. The residual intersection is the required image of C_n.

From K four tangents can be drawn to C_3 besides the one at K. As K describes C_n the four tangents generate a ruled surface R_{2n+6} of order $2n+6$, having C_n for a fourfold curve and C_1 for a multiple line of order $2n+2$, as may be seen as follows: The tangent cone to F_4 from any point P on C_1 consists of the tangent plane at P counted twice and of a cone of order 10 on which C_1 is fourfold. This cone intersects the projecting cone of C_n from P in C_1 counted $4(n-1)$ times and in $6n+4$ other lines. These $6n+4$ lines are composed of two groups; those of the first are the $2n+1$ tangents to C_3 at points of C_n, each counted twice, and the remaining lines belong to the second group. The lines of the second group are generators of the ruled surface in question. The residual intersection of R_{2n+6} with F_4 is a curve C counted twice that remains invariant under the transformation T. It is defined symbolically by the equation

$$2C=(2n+6)(C_1+C_3)-4C_n-(2n+2)C_1,$$

or

$$C=2C_1+(n+3)C_3-2C_n.$$

As K describes C_n its first tangential K' describes a rational curve. By projecting each C_3 from the corresponding K' upon itself, another Cremona transformation T' is defined. Since by T' both $|C_3|$ and C_n remain invariant, and since

$$[\bar{C}_1,\bar{C}_1]=-2,\qquad [\bar{C}_1,\bar{C}_3]=3,\qquad [\bar{C}_1,\bar{C}_n]=n-1,$$

we easily obtain the equations

$$T'.\qquad \bar{C}_1=-C_1+2(n+5)C_3+6C_n,\qquad \bar{C}_3=C_3,\qquad \bar{C}_n=C_n.$$

The determinants of T, T' are each -1, and their product is 1; the operation (TT') is not periodic. Thus the surface is invariant under an infinite number of birational transformations. In any plane the operation (TT')

can be replaced by the product of two projections of the curve upon itself, the center of one of which may be chosen anywhere on the curve. But the operation does not define a continuous group, since the C_3 associated with K has not a constant modulus as K describes C_n.*

Moreover, it can be shown by the same method as that employed by Severi† that all the effective transformations that leave F_4 invariant can be expressed in terms of T and T'.

4. **One double point on the line.** Let all the curves C_3 pass through one fixed point P on C_1, the other points of intersection of C_1, C_3 being variable.

The surface is evidently invariant under the projection of the surface upon itself from the point P; the projection is an involutional Cremona transformation of the Jonquières type. The surface is also invariant under the transformation defined by the first tangential of P as in the preceding case. Surfaces of this type necessarily contain rational curves which cut every plane C_3 of the pencil in one point apart from the point P.

Symbolically, the transformation is expressed as follows. For basis we may choose C_1, C_3 and the point P with the following characteristic‡

$$[C_1, C_1] = -2, \quad [C_1, C_3] = 2, \quad [C_3, C_3] = 0,$$
$$[C_1, P] = 1, \quad [C_3, P] = 1, \quad [P, P] = -2.$$

The point P is a double point on F_4. The tangent cone to F_4 at P is a quadric passing through C_1; the residual intersection is a rational curve of order 7 which passes through P five times. We have therefore

$$[C_7, P] = 5, \quad [C_7, C_1] = 1, \quad [C_7, C_3] = 1, \quad [C_7, C_7] = -2,$$

so that the equations of the projection from P are easily found to be

$$S. \qquad \begin{aligned} \overline{C}_1 &= C_1, \\ \overline{C}_3 &= C_3, \\ \overline{P} &= C_1 + 2C_3 - P. \end{aligned}$$

The equations of the transformation defined by the first tangential are similarly found to be

$$S'. \qquad \begin{aligned} \overline{C}_1 &= -C_1 + 10C_3 + 4P, \\ \overline{C}_3 &= C_3, \\ \overline{P} &= P. \end{aligned}$$

* Picard et Simart, *Fonctions algébriques de deux variables indépendentes*, vol. 2, pp. 517–518.

† Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

‡ For the idea of using a point as part of a basis, see F. Severi, *La base minima pour la totalité des courbes tracés sur une surface algébrique*, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), pp. 449–468.

The product (SS') is not periodic, hence the surface is invariant under a group of infinite order.

5. **Line an inflexional tangent.** The cases in which two distinct or coincident points of intersection of C_3 and C_1 are fixed offer no new peculiarities; similarly for the cases of three distinct fixed points or of two coincident and one distinct fixed points. But the case in which all three points are coincident merits special attention. If C_1 is an inflexional tangent for every C_3 passing through it, the point of inflexion must be the same point for every curve of the pencil, as is easily seen by considering the binary cubic whose roots define the point of contact.

An equation of the form

$$\phi(x_2 : x_3) + mf(x_2 : x_3) = 0$$

must be a cube in $x_2 : x_3$ for every value of m; this is possible only when $\phi \equiv kf$, k being a constant.

The equation of the most general quartic surface Φ_4 passing through a line $(x_1 = 0, x_4 = 0)$ and having the property that every plane section (cubic curve) through the line will have a fixed point of inflexion $(0, 0, 1, 0)$ on the line and the line itself for inflexional tangent, is of the form

$$\Phi_4 \equiv x_1 F_3(x_1, x_2, x_3, x_4) + x_4 \phi_3(x_2, x_3, x_4) = 0,$$

where

$$F_3 = \sum_{i=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4} a_{ikl}\, x_i\, x_k\, x_l, \qquad \phi_3 = \sum_{i=2}^{4}\sum_{k=2}^{4}\sum_{l=2}^{4} b_{ikl}\, x_i\, x_k\, x_l,$$

and

$$a_{223} = a_{233} = a_{333} = b_{223} = b_{233} = b_{333} = 0.$$

The tangent plane to F_4 at any point of the line $x_1 = 0, x_4 = 0$, except the point $(0, 0, 1, 0)$, is $a_{222}\, x_1 + b_{222}\, x_4 = 0$, and is a fixed plane throughout the line. It is no restriction to call this plane $x_1 = 0$, so that $b_{222} = 0$.

The equation of the surface now has the simpler form

$$x_1 F_3 + x_4^2 \phi_2 = 0,$$

where

$$\phi_2 \equiv \sum_{i=2}^{4}\sum_{k=2}^{4} b_{ik}\, x_i\, x_k.$$

The point $(0, 0, 1, 0)$ is a biplanar point, the equations of the two tangent planes being

$$a_{133}\, x_1^2 + a_{334}\, x_1\, x_4 + b_{334}\, x_4^2 = 0.$$

One of these planes may be taken for $x_4 = 0$, by putting $a_{133} = 0$. The surface may be mapped on a double plane $(x_3 = 0)$ by projecting from the double point $(0, 0, 1, 0)$.

The curve of branch points is the intersection of the cone of tangents from the double point. By arranging the equation of the surface in the form

$$u_2 x_3^2 + 2u_3 x_3 + u_4 = 0,$$

where $u_2 = x_4(a_{334} x_1 + b_{334} x_4)$, $u_3 =$ quadratic in x_1, x_4,

$$u_4 = \text{quadratic in } x_1, x_4 + a_{222} x_1 x_2,$$

the equation of the cone may be written in the form

$$u_3^2 - u_2 u_4 \equiv K_6 = p_3 x_2^3 + p_4 x_2^3 + p_5 x_2 + p_6 = 0,$$

where

$$p_3 = a_{222} x_1 x_4 (a_{334} x_1 + b_{33} x_4).$$

If now a curve exists on F_4 which meets every plane cubic through the line $x_1 = 0$, $x_4 = 0$, in one point, it must be rational and meet the line in $n - 1$ points.

Projected from $(0, 0, 1, 0)$, the curve lies on the cone $qx_2 - q' = 0$, in which q, q' are binary in x_1, x_4 of orders $n - 1$, n respectively.

Since every generator of the cone meets Φ_4 in one other point, the locus of this residual point is another rational curve lying on Φ_4. The first necessary condition that $qx_2 - q' = 0$ shall meet Φ_4 in two distinct curves is that Q shall be a contact curve of K_6. If this condition is satisfied, the equation obtained by eliminating x_2 between K_6 and Q has only double roots, so that

$$\frac{p_3 q'^3 + p_4 \cdot q'^2 q + p_5 q' \cdot q^2 + p_6 q^3}{q^3}$$

is of the form I^2/q^3.

These conditions can be satisfied by properly choosing the constants in q' and q. The second condition is that the cone from $(0, 1, 0, 0)$ through the curve of intersection of Φ_4 and Q shall be composite.

By writing the equation of Φ_4 in the form

$$(u_2 x_2 + u_3)^2 = K_6$$

and making it simultaneous with Q, we have

$$q^3 (u_2 x_2 + u_3)^2 = I^2.$$

This will be composite when and only when q is a square, hence a further condition is stated in the following

THEOREM. *If a rational curve exists on Φ_4 cutting every cubic curve through $x_1 = 0$, $x_4 = 0$ in one point, it must be of odd order.*

If q is a square, it can contain only $n/2 - 1$ constants, while q' contains $n + 2$ constants, and $\frac{3}{2}(n + 2)$ relations among the coefficients must be satisfied.

Since

$$n + 2 + \frac{n}{2} - 1 = \frac{3n+2}{2},$$

it follows that there are not enough constants to satisfy the conditions, hence: *The general quartic surface through the line* $x_1 = 0, x_4 = 0$, *having a point of inflexion at* $(0, 1, 0, 0)$ *and* $x_1 = 0, x_4 = 0$ *for inflexional tangent on the cubic curve in every plane through the line, does not contain any rational curve which cuts every cubic curve once.*

The surface does not have an infinite number of birational transformations under which it is invariant, but only the involution defined by projecting Φ_4 upon itself from the point P.

6. **Inflexional tangent obtained by transformation.** More generally, given a surface F_m of order m, having the line $x_1 = 0, x_4 = 0$ for an $(m-3)$-fold line and a rational curve meeting every plane section through the line once, it is possible to transform the surface into a surface of order m' having an $(m'-3)$-fold straight line on it, and having the further property that every plane cubic section through the line will have the line for inflexional tangent.

At any point P of the rational curve the tangent t and any other line n through P can be chosen in terms of which the equation of the cubic may be expressed in the form

$$lt - n^2 + \cdots = 0, \qquad l(p) \neq 0.$$

The conic $lt - n^2 = 0$ has three point contact with the cubic at P, and meets it in three other points P_1, P_2, P_3. By using the triangle $P_1 P_2 P_3$ for triangle of quadratic inversion, the cubic will go over into a cubic and the conic into a straight line having at the point P_1' (the image of P) three point contact with the image cubic. Since a net of conics can be drawn having three point contact with the cubic at P, there are two independent parameters, enough to make the image line any line in the plane. The coördinates of P_1, P_2, P_3 enter the equations of transformation symmetrically, and all the operations can be expressed rationally.

But it has been seen that, if every C_3 has C_1 for inflexional tangent, the point of contact of C_3 with C_1 is fixed.

It is thus shown that a surface containing a pencil of plane cubic curves and a rational curve cutting each curve of the pencil once can be transformed into another having the given line for inflexional tangent of every plane cubic section through it. The converse, however, is not true. In the transformed surface sufficient restrictions are imposed on the coefficients to permit the existence of a rational curve, and hence of an infinite number. These curves are the transforms of the locus of the successive tangentials of P on the original surface.

2. Quartic Surfaces Containing a Conic

7. F_4 with two coplanar conics. If one plane section of a quartic surface is composed of two conics, the number of linearly independent curves on the surface is two.*

Since neither conic is a multiple of the other, the two conics C_2 and C_2' form a basis. The surface is regular and of geometric genus 1, and hence the grade n and the genus π of any curve C on the surface satisfy the relation

$$n = 2\pi - 2.$$

The curves C are self-adjoint and of dimensions† π. In the preceding cases the question of the existence of a birational transformation did not need to be considered, as each transformation could be determined by geometric considerations. In the present case the existence is not so evident, and we shall proceed algebraically.

Since the conics C_2, C_2' are rational and meet in four points, any two curves $\lambda C_2 + \mu C_2'$, $\lambda' C_2 + \mu' C_2'$ meet in i points, defined by

$$i = -2\lambda\lambda' + 4(\lambda\mu' + \lambda'\mu) - 2\mu\mu',$$

and the grade n of $\lambda C_2 + \mu C_2'$ is given by

$$n = -2\lambda^2 + 8\lambda\mu - 2\mu^2.‡$$

Hence $\lambda C_2 + \mu C_2'$, which is of order $2\lambda + 2\mu$, meets C_2 in $4\mu - 2\lambda$ points and meets C_2' in $4\lambda - 2\mu$ points. It follows that λ and μ are positive integers and that neither can exceed twice the other except the cases $\lambda = 1$, $\mu = 0$ and $\lambda = 0$, $\mu = 1$. We shall now show that C_2, C_2' form a minimum basis§ so that all curves on F_4 are expressible in the form $\lambda C_2 + \mu C_2'$.

Suppose Γ and Γ' constitute a minimum basis. Then

$$C_2 = \alpha\Gamma + \beta\Gamma',$$

$$C_2' = \gamma\Gamma + \delta\Gamma',$$

in which α, β, γ, δ are integers. If $|\Gamma|$ is of grade a, $|\Gamma'|$ of grade b, and any curve $|\Gamma|$ meets $|\Gamma'|$ in h points, the grade of any curve $l\Gamma + m\Gamma'$ is

$$al^2 + 2hlm + bm^2.$$

* Bagnera and de Franchis, *Le nombre ρ de M. Picard pour les surfaces hyperelliptiques*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 185–238. See page 187.

† F. Severi, *Complementi alla theoria delle base per la totalità delle curve di una superfice algebrica*, Rendiconti del circolo matematico di Palermo, vol. 30 (1911), pp. 265–288, p. 276.

‡ F. Severi, Mathematische Annalen, loc. cit., see p. 222.

§ F. Severi, Rendiconti di Palermo, loc. cit., p. 277.

Since $\lambda C_2 + \mu C_2'$ may be written in the form

$$(\lambda\alpha + \mu\gamma)\,\Gamma + (\lambda\beta + \mu\delta)\,\Gamma',$$

we have, by making this identical with $l\Gamma + m\Gamma'$,

$$a(\alpha\lambda + \gamma\mu)^2 + 2h(\alpha\lambda + \gamma\mu)(\beta\lambda + \delta\mu) + b(\beta\lambda + \delta\mu)^2 = -2\lambda^2 + 8\lambda\mu - 2\mu^2.$$

Hence

$$(ab - h^2)(\alpha\delta - \beta\gamma)^2 = -12.$$

Since a, b, and h are integers, it follows that $\alpha\delta - \beta\gamma = \pm 1$ or ± 2. In the former case C_2 and C_2' also form a minimum basis. In any case a and b are even, hence the value ± 2 is impossible, since it requires h^2 to be of the form $4p + 3$, in which p is an integer. We have therefore proved that the conics C_2 and C_2' form a minimum basis. The following table of possible curves on F_4 can now be found.

λ or μ	μ or λ	Order	Genus
1	0	2	0
1	1	4	3
2	1	6	4
2	2	8	9

The curves $\lambda C_2 + \mu C_2'$, $\lambda \geqq \mu$, are the residual intersections of F_4 with surfaces of order λ having C_2' for a $(\lambda - \mu)$-fold curve. All the formulas can be readily verified by the Cayley-Noether equivalence and postulation theorems.* Hence the conics C_2 and C_2' are the only rational curves on the surface; the curves of genus two are composed of the partial series of tangent planes; the only curves of genus three are the plane sections $|C_2 + C_2'|$, etc.

If a birational transformation exists that leaves F_4 invariant, every linear system of curves is transformed into another linear system of the same grade, genus, and dimension. Hence the plane sections must be transformed into plane sections and the two conics either interchanged or left invariant. This is possible only under a linear transformation, and the general quartic surface containing a conic does not admit such a transformation. We may therefore conclude: *The most general quartic surface subjected only to the condition that it must contain a conic is not invariant under any birational transformation.*†

* M. Noether, *Sulle curve multiple di superficie algebriche*, Annali di matematica, ser. 2, vol. 5 (1871), pp. 163–177. See p. 164 and 168.

† L. Godeaux, *Sur la surface du quatrième ordre contenant une conique*, Société des sciences du Hainaut, vol. 62 (1912), pp. 1–8, has considered the problem of possible birational transformations of the quartic surface containing a conic. In consequence of the restriction $\lambda \geqslant 2\mu$ (in our notation) which he overlooks, the system of rational curves there found are all composite. The proof of the non-existence of involutional transformations is also incomplete because not all possible cases are considered.

3. Quartics Containing One or Two Space Cubic Curves

8. **F_4 through a space cubic curve.** Let F_4 contain a space cubic curve C_3. From any point P on F_4 can be drawn one bisecant to C_3 which meets F_4 in a fourth point P'. Since the bisecant to C_3 from P' meets F_4 in P, we see that F_4 admits an involutional birational transformation. Through C_3 pass a bundle of quadric surfaces, each meeting F_4 in a residual quintic of genus 2. All the quadrics of the bundle which pass through a fixed point P belong to a pencil whose basis curve consists of C_3 and the bisecant through P. Hence all the quintic curves through P on F_4 also pass through P'.

A basis is composed of C_3 and a plane section C_4. By proceeding as in the preceding cases we obtain the transformation

$$U. \qquad \begin{aligned} \overline{C}_4 &= 9C_4 - 5C_3, \\ \overline{C}_3 &= 16C_4 - 9C_3, \end{aligned}$$

which can be verified as follows:

The image of a plane section C_4 is obtained by constructing the ruled surface of bisecants to C_3 from each point of C_4. The curve C_4 is a simple curve on the surface, and the lines joining the three points of intersection of C_3 and C_4 in pairs are double generators. Hence the surface is of order 10. From any point of C_3 a quadric cone of lines can be drawn meeting C_3 again. This cone meets the quartic cone to C_4 in eight lines, but three of them pass through the points of intersection of C_3 and C_4; hence C_3 is a five-fold curve on the ruled surface. Since $\overline{C}_4$ is the residual intersection of F_4 and the ruled surface, we have

$$\overline{C}_4 = 10C_4 - (C_4 + 5C_3).$$

The image of C_3 is obtained as the residual intersection with F_4 of the ruled surface obtained by connecting each point K of C_3 with the point K' in which the tangent plane to F_4 at K meets C_3 again.

From K can be drawn one generator to K'; from K' can be drawn six generators to points of tangency K. Moreover, the ruled surface and F_4 touch each other at every point K of C_3; hence each generator meets 14 others, and the surface is of order 16. The curve C_3 counts for nine intersections, and hence the residual intersection C_3 is defined by the formula

$$\overline{C}_3 = 16C_4 - 9C_3.$$

The other possible values of λ and μ which satisfy the congruences analogous to those of § 2 lead to non-effective curves. Hence the surface does not admit an infinite number of birational transformations.

The transformation is a Cremona transformation, the invariant surface

being the locus of the double points of the involution determined on C_3 and F_4 by the bisecants of C_3.

9. **F_4 through two space cubic curves.** Consider the general quartic surface containing the two space cubic curves C_3 and C_3' which intersect in k points $0 \leqq k \leqq 5$. A basis may be made of C_3, C_3' and a plane section C_4. The involution defined by transforming P into the residual point P' of intersection with the surface of the bisecant of C_3 through P may be expressed by the equations

$$V. \qquad \begin{aligned} \overline{C}_3 &= 16C_4 - 9C_3, \\ \overline{C}_3' &= (12 - 2k)C_4 - (6 - k)C_3 - C_3', \\ \overline{C}_4 &= 9C_4 - 5C_3. \end{aligned}$$

The similar involution as to C_3' has the equations

$$V'. \qquad \begin{aligned} \overline{C}_3 &= (12 - 2k)C_4 - C_3 - (6 - k)C_3', \\ \overline{C}_3' &= 16C_4 - 9C_3', \\ \overline{C}_4 &= 9C_4 - 5C_3'. \end{aligned}$$

The product of these two involutions is a discontinuous birational transformation that is not periodic.

When $k = 5$, the curves C_3 and C_3' lie on a quadric which meets F_4 in a conic C_2. By the transformation V the curves C_3' and C_2 are interchanged.

Similarly, the curves C_3 and C_2 are interchanged by the transformation V'.

CORNELL UNIVERSITY,
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ONE-PARAMETER FAMILIES OF CURVES IN THE PLANE*

BY

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INTRODUCTION

Nets of curves in the plane were studied from a projective point of view for the first time by Wilczynski.† As he himself points out,‡ the theory as set up by him is that of the "two component one-parameter families, rather than a theory of the net itself." In the present paper, we shall show how a single one-parameter family of plane curves may be studied by Wilczynski's methods. Although many of the projective properties of a single one-parameter family may be investigated by considering it as a component family of a particular net,—such properties are in fact noted in Wilczynski's memoir—still the theory of one of the families of the net is closely bound up with the properties of the other, if the study be made as in the paper already cited. To make this clearer, we must outline Wilczynski's procedure.

If $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ be interpreted as the homogeneous coördinates of a point in the plane, the equations

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3) \tag{1}$$

define for u = const. and v = const. two one-parameter families of curves. Equations (1) are a fundamental system of solutions of the completely integrable system of partial differential equations

$$\begin{aligned} y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= a'y_u + b'y_v + c'y, \\ y_{vv} &= a''y_u + b''y_v + c''y, \end{aligned} \tag{2}$$

any fundamental system of solutions of which, say $\bar{y}^{(1)}$, $\bar{y}^{(2)}$, $\bar{y}^{(3)}$, will be of the form

$$\bar{y}^{(k)} = c_{k1}y^{(1)} + c_{k2}y^{(2)} + c_{k3}y^{(3)}, \qquad |c_{kl}| \neq 0 \quad (k = 1, 2, 3)$$

where the c's are constants. The differential equations therefore define the

* Presented to the Society October 25, 1913.

† *One-parameter families and nets of plane curves.* These Transactions, vol. 12 (1911), pp. 473–510.

‡ Loc. cit., p. 478.

configuration (1) and all its projective transformations. The most general point transformations which leave unchanged the configuration (1) are composed of transformations of the form

$$\bar{y} = \lambda(u, v)\, y,$$

which leave every point of the plane fixed, and

$$\bar{u} = U(u), \qquad \bar{v} = V(v),$$

which leave invariant each of the families u = const., v = const. Suppose these transformations to be made on the differential equations (2). Those properties of (2) which remain unchanged under these transformations will be characteristic of the net (1) and of all its projective transformations, i. e., they will be projective properties. *Invariants* are those functions of the coefficients of (2), and of their derivatives, which remain unchanged under these transformations, except possibly for a factor depending only on the transformations. Covariants have the same nature and properties, but contain also the variable y and its derivatives.

Now evidently the projective properties of the net, as expressed by the vanishing of invariants thus found, can not all be characteristic of a single family of curves of the net, say the family u = const. For instance, the condition that the curves v = const. be straight lines is obviously from the first of equations (2) the vanishing of the invariant b; this has, however, nothing whatever to do with the properties of the curves u = const. But the invariant equation $a'' = 0$ is the condition that the curves u = const. be straight lines, and is therefore characteristic for that family.

Again, the covariants of the net have the following geometric interpretation. The tangents to the curves of one family, constructed at the points where these curves meet a fixed curve of the other family, have a covariant curve as envelope. Evidently neither of the two covariants thus obtained is intrinsically connected with the single family of curves u = const.

The theory of the net of curves, as developed by Wilczynski, is therefore not applicable in its present form to the study of a single family of plane curves, say the curves u = const. defined by (1). We shall in the sequel modify his theory so that it may be used for the theory of a one-parameter family of curves, by making this family one of the components of a net, in which the second family bears an intrinsic geometric relation to the given family.

1. The One-parameter Family of Curves as Part of a Canonical Net

The most general transformation of the independent variables which leaves fixed the family of curves u = const. defined by equations (1) is

$$\bar{u} = U(u), \qquad \bar{v} = V(u, v). \tag{3}$$

By means of this transformation we can change in the most arbitrary way the system of curves $v = \text{const.}$ which cut the curves $u = \text{const.}$ We seek a particular transformation of form (3) which will associate with the family $u = \text{const.}$ a family $v = \text{const.}$ intrinsically related to it. For a family of space curves which do not lie on a developable surface and which are not asymptotic curves on the surface determined by them, there is a very evident choice of a second family, namely the conjugate system of curves.* For the system of plane curves there are of course isogonal trajectories, but isogonality is not a projective property. There is in fact no obvious projective relation, analogous say to the relation of conjugacy of curves on a surface, of which we can make use here. We proceed to determine a family of curves $v = \text{const.}$ which have a projectively intrinsic relation to the given family $u = \text{const.}$, but the relation is by no means so simple as that of conjugacy on a curved surface.

Let us carry out the transformation

$$\bar{u} = u, \qquad \bar{v} = \phi(u, v), \tag{4}$$

which is included in (3), on the system of differential equations. We have

$$\begin{aligned} y_u &= \bar{y}_u + \bar{y}_v \phi_u, \qquad y_v = \bar{y}_v \phi_v, \\ y_{uu} &= \bar{y}_{uu} + 2\bar{y}_{uv}\phi_u + \bar{y}_{vv}\phi_u^2 + \bar{y}_u + \bar{y}_v \phi_{uu}, \\ y_{uv} &= \bar{y}_{uv}\phi_v + \bar{y}_{vv}\phi_u\phi_v + \bar{y}_v \phi_{uv}, \\ y_{vv} &= \bar{y}_{vv}\phi_v^2 + \bar{y}_v \phi_{vv}, \end{aligned} \tag{5}$$

where we have written $\bar{y}_u$ for $\partial y/\partial \bar{u}$, $\bar{y}_v$ for $\partial y/\partial \bar{v}$, etc. Substituting these equations in (2), and solving for the new second derivatives $\bar{y}_{uu}$, $\bar{y}_{uv}$, $\bar{y}_{vv}$, we obtain a system of differential equations of the same form as (2), with coefficients $\bar{a}$, $\bar{b}$, etc., given by

$$\begin{aligned} \bar{a} &= a - 2a'\frac{\phi_u}{\phi_v} + a''\frac{\phi_u^2}{\phi_v^2}, \\ \bar{b} &= (a - 2b')\phi_u + b\phi_v + (b'' - 2a')\frac{\phi_u^2}{\phi_v} + a''\frac{\phi_u^3}{\phi_v^2} \\ &\qquad - \phi_{uu} + 2\frac{\phi_u}{\phi_v}\phi_{uv} - \frac{\phi_u^2}{\phi_v^2}\phi_{vv}, \\ \bar{c} &= c - 2c'\frac{\phi_u}{\phi_v} + c''\frac{\phi_u^2}{\phi_v^2}, \\ \phi_v \bar{a}' &= a' - a''\frac{\phi_u}{\phi_v}, \qquad \phi_v \bar{c}' = c' - c''\frac{\phi_u}{\phi_v}, \end{aligned} \tag{6}$$

* The author has already carried out this idea in a paper entitled *Projective Differential Geometry of One-parameter Families of Space Curves, and Conjugate Nets on a Curved Surface*, to appear in the American Journal of Mathematics.

$$\bar{b}' = b' + (a' - b'')\frac{\phi_u}{\phi_v} - a''\frac{\phi_u^2}{\phi_v^2} - \frac{\phi_{uv}}{\phi_v} + \frac{\phi_u}{\phi_v}\frac{\phi_{vv}}{\phi_v},$$

$$\phi_v^2\,\bar{a}'' = a'', \qquad \phi_v\,\bar{b}'' = b'' + a''\frac{\phi_u}{\phi_v} - \frac{\phi_{vv}}{\phi_v}, \qquad \phi_v^2\,\bar{c}'' = c''.$$

Let us put

$$\mathfrak{A}' = a' - \tfrac{1}{3}(a' + b'') + \frac{1}{6}\frac{a_v''}{a''},$$

and calculate the corresponding quantity $\overline{\mathfrak{A}}'$. We have

$$\frac{\bar{a}_v''}{\bar{a}''} = \frac{\partial}{\partial \bar{v}}\log \bar{a}'' = \frac{1}{\phi_v}\frac{\partial}{\partial v}\log \bar{a}'' = \frac{1}{\phi_v}\left(\frac{a_v''}{a''} - 2\frac{\phi_{vv}}{\phi_v}\right),$$

and

$$\begin{aligned}\phi_v\,\overline{\mathfrak{A}}' &= \phi_v\left[\bar{a}' - \tfrac{1}{3}(\bar{a}' + \bar{b}'') + \frac{1}{6}\frac{\bar{a}_v''}{\bar{a}''}\right] \\ &= a' - \tfrac{1}{3}(a' + b'') + \frac{1}{6}\frac{a_v''}{a''} - a''\frac{\phi_u}{\phi_v} \\ &= \mathfrak{A}' - a''\frac{\phi_u}{\phi_v}.\end{aligned} \tag{7}$$

We may therefore make $\overline{\mathfrak{A}}'$ vanish by taking for ϕ a solution of the partial differential equation of the first order

$$a''\,\phi_u - \mathfrak{A}'\,\phi_v = 0. \tag{8}$$

This choice of $\phi(u, v)$ gives a uniquely determined family $\bar{v} = \text{const.}$ For, from (7) we see that the most general transformation of the independent variable v which leaves the condition $\mathfrak{A}' = 0$ unchanged is independent of u; also, the condition will be found to remain invariant under any transformation $\bar{u} = U(u)$. Consequently the family $\bar{v} = \text{const.}$ bears a characteristic relation to the family $\bar{u} = \text{const.}$, of such nature, that it remains undisturbed only by those transformations of the independent variables which are of the form

$$\bar{u} = U(\bar{u}), \qquad \bar{v} = V(\bar{v}).$$

We can make this clearer by inquiring into the geometric meaning of the relation $\mathfrak{A}' = 0$. This condition has been interpreted by Wilczynski.* We shall give here an adequate description of the geometric nature of the net, but must refer to Wilczynski's memoir for the demonstrations.

Consider any net $u = \text{const.}$, $v = \text{const.}$ At each point of a fixed curve $u = \text{const.}$ construct the tangent to the curve $v = \text{const.}$ which passes through that point. The ∞^1 tangents so drawn will envelop a curve. The family of

* Loc. cit., p. 504.

curves corresponding in this way to the family $u = \text{const.}$ is given by the equations

$$\rho^{(k)} = y_u^{(k)} - b'\, y^{(k)} \qquad (k = 1, 2, 3),$$

or, as we may say, by the covariant (of the net)

$$\rho = y_u - b'\, y,$$

for $u = \text{const.}$* Thus to every point P_y given by the equations

$$y^{(k)} = f^{(k)}(u, v)$$

corresponds a covariant point P_ρ. In the same way, a covariant

$$\sigma = y_v - a'\, y$$

gives points P_σ, which for $v = \text{const.}$ trace the envelope of the tangents to the curves $u = \text{const.}$ at the points where these curves meet the fixed curve $v = \text{const.}$ given by y. We have then at any point P_y two lines, tangent respectively to the curves $u = \text{const.}$ and $v = \text{const.}$ which pass through P_y, and on these two lines the respective covariant points P_σ, P_ρ.

Now construct the conic which osculates the curve $u = \text{const.}$ at P_y. *The line $P_\sigma P_\rho$ cuts it in a pair of points, which are harmonic to the pair P_ρ, P_σ if and only if* $\mathfrak{A}' = 0$.†

Let us call the net so determined the *canonical net.* The covariants

$$\bar\rho = \bar y_u - \bar b'\, \bar y, \qquad \bar\sigma = \bar y_v - \bar a'\, \bar y \tag{9}$$

will give for each point P_y two points $P_{\bar\rho}$ and $P_{\bar\sigma}$, and the line $P_y P_{\bar\rho}$ will be the polar of $P_{\bar\sigma}$ with respect to the conic which osculates the curve $u = \text{const.}$ at P_y. We proceed now to show geometrically how the family $u = \text{const.}$ determines the second family, $\bar v = \text{const.}$, of the canonical net. Of course, the best we can do is to find at each point P_y the direction of the line $P_y P_{\bar\rho}$, since the determination of the family $\bar v = \text{const.}$ itself requires the integration of equation (8).

Suppose known only the family of curves $u = \text{const.}$ Draw the tangent t and the osculating conic C at P_y. Consider any line l through P_y. Its pole with respect to the conic C is a point Q on t. The tangents to the curves $u = \text{const.}$ at the points where these curves meet l envelop a curve; let R be the point of this curve which lies on t. Consider now the pencil of lines $[l]$ through P_y. The corresponding poles will be a range $[Q]$ on t; corresponding to each Q will be a point R given by the construction above. The ranges $[Q]$ and $[R]$ are in one to one correspondence. There are two double-points $Q \equiv R$; one of them is the point P_y, the other a definite point which is in

* E. J. Wilczynski, loc. cit., p. 486.

† E. J. Wilczynski, loc. cit., p. 504.

fact the covariant point $P_{\bar\sigma}$. The polar of $P_{\bar\sigma}$ with respect to C gives a certain direction passing through P_y. This will give the direction of $P_y P_{\bar\rho}$, although the point $P_{\bar\rho}$ is not thereby determined. It is possible, therefore, to construct at each point P_y the direction $P_y P_{\bar\rho}$; the ∞^2 directions so constructed give on integration the second family, $\bar v$ = const., of the canonical net.

Since the canonical net is completely determined by the given family of curves u = const., it follows that the invariants and covariants of the net are invariants and covariants of the given family of curves. Thus the quantities $\bar\rho$, $\bar\sigma$ given by (9) are covariants of the family u = const. The invariants may be calculated as in Wilczynski's memoir. Put

$$\bar f_u = \bar a + \bar b', \qquad \bar f_v = \bar a' + \bar b'',$$

$$\bar A = \bar a - \tfrac{2}{3}\bar f_u, \qquad \bar B = \bar b, \qquad \bar C = \bar c + \tfrac{1}{3}\bar a \bar f_u + \tfrac{1}{3}\bar b \bar f_v - \tfrac{1}{3}\bar f_{uu} - \tfrac{1}{9}\bar f_u^2, \tag{10}$$

$$\bar A' = \bar a' - \tfrac{1}{3}\bar f_v, \qquad \bar B' = \bar b' - \tfrac{1}{3}\bar f_u,$$

$$\bar C' = \bar c' + \tfrac{1}{3}\bar a' \bar f_u + \tfrac{1}{3}\bar b' \bar f_v - \tfrac{1}{3}\bar f_{uv} - \tfrac{1}{9}\bar f_u \bar f_v, \tag{11}$$

$$\bar A'' = \bar a'', \qquad \bar B'' = \bar b'' - \tfrac{2}{3}\bar f_v,$$

$$\bar C'' = \bar c'' + \tfrac{1}{3}\bar a'' \bar f_u + \tfrac{1}{3}\bar b'' \bar f_v - \tfrac{1}{3}\bar f_{vv} - \tfrac{1}{9}\bar f_v^2.$$

Then every invariant of the canonical net, and hence of the family of curves u = const., is a function of the following fundamental system of invariants and their derivatives:*

$$\bar{\mathfrak{B}} = \bar B, \qquad \bar{\mathfrak{C}} = \bar C - \bar B'_u - 2\bar B'^2 + \bar A' \bar B,$$

$$\bar{\mathfrak{A}}' = \bar A' + \frac{1}{6}\frac{\bar A''_v}{\bar A''}, \qquad \bar{\mathfrak{B}}' = \bar B' + \frac{1}{6}\frac{\bar B_u}{\bar B}, \qquad \bar{\mathfrak{C}}' = \bar C' + \bar A' \bar B', \tag{12}$$

$$\bar{\mathfrak{A}}'' = \bar A'', \qquad \bar{\mathfrak{C}}'' = \bar C'' - \bar A'_v - 2\bar A'^2 + \bar A'' \bar B'.$$

The last statement is evidently true from a geometric point of view; analytically, it may be verified as follows. In equations (6) appear the first and second derivatives of ϕ. From (8) we obtain by differentiation the equations

$$\phi_{uu} - \frac{\phi_u}{\phi_v}\phi_{uv} = \frac{\mathfrak{A}'_u}{a''}\phi_v - \frac{a''_u}{a''}\phi_u,$$

$$\phi_{uv} - \frac{\phi_u}{\phi_v}\phi_{vv} = \frac{\mathfrak{A}'_v}{a''}\phi_v - \frac{a''_v}{a''}\phi_u. \tag{13}$$

By means of these and (8) we may eliminate from expressions (6) all derivatives of ϕ but one, say ϕ_v, except in the expression for $\bar b''$, from which ϕ_v and ϕ_{vv} cannot be removed. In all of the expressions thus obtained ϕ_v comes

* The invariant $\bar{\mathfrak{A}}' = \bar A' + \frac{1}{6}\frac{\bar A''_v}{\bar A''}$ is of course identically equal to zero.

out as a factor; moreover, in the invariants (12) $\bar{b}''$ occurs only in combinations from which ϕ_{vv} disappears. It follows upon calculation that *except for extraneous factors (powers of* ϕ_v*), the invariants* (12) *are expressible rationally in terms of the coefficients of the original system of differential equations* (2)*, and the derivatives of these coefficients.*

The covariants (9) are expressible in the same way; we have, in fact,

$$\bar{y}_u = y_u - \frac{\phi_u}{\phi_v} y_v, \qquad \bar{y}_v = \frac{1}{\phi_v} y_v,$$

so that on calculating the values of $\bar{a}'$ and $\bar{b}'$ we obtain

$$\bar{\rho} = y_u - \frac{\mathfrak{A}'}{a''} y_v - \left[b' + \frac{\mathfrak{A}'}{a''} \left(a' - b'' - \mathfrak{A}' + \frac{a_v''}{a''} - \frac{\mathfrak{A}_v'}{\mathfrak{A}'} \right) \right] y, \tag{14}$$

$$\phi_v \bar{\sigma} = y_v - (a' - \mathfrak{A}')\, y.$$

All covariants of the canonical net are functions of these, of y, and of invariants; it may be shown, moreover, that the complete system of invariants, as set up by Wilczynski, is expressible entirely in terms of the coefficients of (2). We may state our results as follows: *the projective geometry of a one-parameter family of plane curves may be investigated by the study of a canonical net whose determination requires the integration of a partial differential equation of the first order. This integration is however unnecessary for the formulation of the projective properties of the one-parameter family of curves, since the invariants and covariants of this geometric configuration are expressible entirely in terms of the coefficients and variables of the original system of differential equations which define the one-parameter family as one component family of a net in which the second component family is arbitrary.*

2. Canonical Development for a One-parameter Family of Plane Curves

Suppose the family $v = \text{const.}$ has been determined so that it forms with the given system $u = \text{const.}$ a canonical net. We have then

$$\bar{y}_u = \bar{\rho} + \bar{b}'\, \bar{y}, \qquad \bar{y}_v = \bar{\sigma} + \bar{a}'\, \bar{y}, \tag{15a}$$

$$\bar{y}_{uu} = \bar{a}\bar{y}_u + \bar{b}\bar{y}_v + \bar{c}\bar{y},$$

$$\bar{y}_{uv} = \bar{a}'\, \bar{y}_u + \bar{b}'\, \bar{y}_v + \bar{c}'\, \bar{y}, \tag{15b}$$

$$\bar{y}_{vv} = \bar{a}''\, \bar{y}_u + \bar{b}''\, \bar{y}_v + \bar{c}''\, \bar{y}.$$

The second derivatives of $\bar{y}$ are therefore expressible linearly in terms of $\bar{\rho}, \bar{\sigma}, \bar{y}$; the same is easily seen to be true of all higher derivatives of $\bar{y}$. We have, then, in general,

$$\frac{\partial^{i+j}}{\partial \bar{u}^i \partial \bar{v}^j} \bar{y} = \alpha^{(ij)} \bar{\rho} + \beta^{(ij)} \bar{\sigma} + \gamma^{(ij)} \bar{y}, \tag{15c}$$

where the α's, β's, and γ's are functions of the coefficients in (15b) and of their derivatives. Thus

$$\alpha^{(10)} = 1, \quad \beta^{(10)} = 0, \quad \gamma^{(10)} = \bar{b}',$$
$$\alpha^{(01)} = 0, \quad \beta^{(01)} = 1, \quad \gamma^{(01)} = \bar{a}',$$

from (15a). We may call these "coefficients of the first order." The coefficients of the second order are

$$\begin{aligned} &\alpha^{(20)} = \bar{a}, && \beta^{(20)} = \bar{b}, && \gamma^{(20)} = \bar{c} + \bar{a}\bar{b}' + \bar{b}\bar{a}', \\ &\alpha^{(11)} = \bar{a}', && \beta^{(11)} = \bar{b}', && \gamma^{(11)} = \bar{c}' + 2\bar{a}'\bar{b}', \\ &\alpha^{(02)} = \bar{a}'', && \beta^{(02)} = \bar{b}'', && \gamma^{(02)} = \bar{c}'' + \bar{a}''\bar{b}' + \bar{b}''\bar{a}'. \end{aligned} \tag{16}$$

Some of the coefficients of the third and higher orders are expressible in more than one way; this results from the fact that an expression like

$$\alpha^{(21)} \bar{\rho} + \beta^{(21)} \bar{\sigma} + \gamma^{(21)} \bar{y}$$

may be found either from $\partial \bar{y}_{uu}/\partial \bar{v}$ or from $\partial \bar{y}_{uv}/\partial \bar{u}$. We suppose that in every case two or more expressions for the same coefficient are identically equal* (for all values of $\bar{u}$ and $\bar{v}$). Thus, we have

$$\alpha^{(21)} = \bar{a}_u' + \bar{a}\bar{a}' + \bar{a}'\bar{b}' + \bar{c}' = \bar{a}_v + \bar{a}\bar{a}' + \bar{b}\bar{a}'',$$

etc. There is no theoretical difficulty in the calculation of these coefficients, which we leave for the present.

Let $P_{\bar{y}}$ be any point of the plane. We may suppose without loss of generality that for $P_{\bar{y}}$ the parameters $\bar{u}$, $\bar{v}$ have the values 0, 0. Assume that about $P_{\bar{y}}$ the geometric configuration is regular, so that for a sufficiently small neighborhood about $P_{\bar{y}}$ we have the expansion

$$\mathrm{Y} = \bar{y} + \bar{y}_u \bar{u} + \bar{y}_v \bar{v} + \tfrac{1}{2}(\bar{y}_{uu} \bar{u}^2 + 2\bar{y}_{uv} \bar{u}\bar{v} + \bar{y}_{vv} \bar{v}^2) + \cdots,$$

where Y corresponds to general values of the parameters $\bar{u}$, $\bar{v}$. By using equations (15) we find, therefore,

$$\mathrm{Y} = y_1 \bar{\rho} + y_2 \bar{\sigma} + y_3 \bar{y}, \tag{17}$$

where y_1, y_2, y_3 are the following power series in $\bar{u}$, $\bar{v}$:

* The equality of the pairs of expressions which represent the six coefficients $\alpha^{(21)}$, $\beta^{(21)}$, $\gamma^{(21)}$, $\alpha^{(12)}$, $\beta^{(12)}$, $\gamma^{(12)}$ is sufficient to ensure the equality of ambiguous expressions for all other coefficients. In fact, the six equations expressing the uniqueness in value of $\alpha^{(21)}$, $\beta^{(21)}$, etc., are the conditions of complete integrability for the system of differential equations (15b).

$$
\begin{aligned}
y_1 &= \bar{u} + \tfrac{1}{2}\bar{a}\bar{u}^2 + \bar{a}'\,\bar{u}\bar{v} + \tfrac{1}{2}\bar{a}''\,\bar{v}^2 \\
&\quad + \tfrac{1}{6}\alpha^{(30)}\,\bar{u}^3 + \tfrac{1}{2}\alpha^{(21)}\,\bar{u}^2\,\bar{v} + \tfrac{1}{2}\alpha^{(12)}\,\bar{u}\bar{v}^2 + \tfrac{1}{6}\alpha^{(03)}\,\bar{v}^{(3)} \\
&\quad + \tfrac{1}{24}\alpha^{(40)}\,\bar{u}^4 + \tfrac{1}{6}\alpha^{(31)}\,\bar{u}^3\,\bar{v} + \tfrac{1}{4}\alpha^{(22)}\,\bar{u}^2\,\bar{v}^2 + \tfrac{1}{6}\alpha^{(13)}\,\bar{u}\bar{v}^3 + \tfrac{1}{24}\alpha^{(04)}\,\bar{v}^4 \\
&\quad + \cdots, \\
y_2 &= \bar{v} + \tfrac{1}{2}\bar{b}\bar{u}^2 + \bar{b}'\,\bar{u}\bar{v} + \tfrac{1}{2}\bar{b}''\,\bar{v}^2 \\
&\quad + \tfrac{1}{6}\beta^{(30)}\,\bar{u}^3 + \tfrac{1}{2}\beta^{(21)}\,\bar{u}^2\,\bar{v} + \tfrac{1}{2}\beta^{(12)}\,\bar{u}\bar{v}^2 + \tfrac{1}{6}\beta^{(03)}\,\bar{v}^3 \\
&\quad + \tfrac{1}{24}\beta^{(40)}\,\bar{u}^4 + \tfrac{1}{6}\beta^{(31)}\,\bar{u}^3\,\bar{v} + \tfrac{1}{4}\beta^{(22)}\,\bar{u}^2\,\bar{v}^2 + \tfrac{1}{6}\beta^{(13)}\,\bar{u}\bar{v}^3 \\
&\quad + \tfrac{1}{24}\beta^{(04)}\,\bar{v}^4 + \cdots, \\
y_3 &= 1 + \bar{b}'\,\bar{u} + \bar{a}'\,\bar{v} + \tfrac{1}{2}\gamma^{(20)}\,\bar{u}^2 + \gamma^{(11)}\,\bar{u}\bar{v} + \tfrac{1}{2}\gamma^{(02)}\,\bar{v}^2 \\
&\quad + \tfrac{1}{6}\gamma^{(30)}\,\bar{u}^3 + \tfrac{1}{2}\gamma^{(21)}\,\bar{u}^2\,\bar{v} + \tfrac{1}{2}\gamma^{(12)}\,\bar{u}\bar{v}^2 + \tfrac{1}{6}\gamma^{(03)}\,\bar{v}^3 \\
&\quad + \cdots,
\end{aligned}
\tag{18}
$$

in which terms omitted are in each case of higher order than the last term written.

Except for isolated points $\bar{y}$ of the plane, the covariant points $\bar{\rho}$, $\bar{\sigma}$ which correspond to the point $\bar{y}$ form with $\bar{y}$ a non-degenerate triangle, which we shall take as a moving triangle of reference. A point whose three coördinates are

$$x^{(k)} = x_1\,\bar{\rho}^{(k)} + x_2\,\bar{\sigma}^{(k)} + x_3\,\bar{y}^{(k)} \qquad (k = 1, 2, 3),$$

or, as we may say, a point given by

$$x = x_1\,\bar{\rho} + x_2\,\bar{\sigma} + x_3\,\bar{y},$$

will have coördinates x_1, x_2, x_3 when referred to the covariant triangle $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$, $P_{\bar{y}}$, if the unit point of the new coördinate system be properly chosen. Hence the point Y, given by (17), has coördinates y_1, y_2, y_3 when referred to the triangle $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$, $P_{\bar{y}}$.

We now introduce non-homogeneous coördinates ξ, η, by putting

$$\xi = \frac{y_1}{y_3}, \qquad \eta = \frac{y_2}{y_3}.$$

We have, to terms of the third order,

$$
\begin{aligned}
\frac{1}{y_3} &= 1 - \bar{b}'\,\bar{u} - \bar{a}'\,\bar{v} + C^{(20)}\,\bar{u}^2 + C^{(11)}\,\bar{u}\bar{v} + C^{(02)}\,\bar{v}^2 \\
&\quad + C^{(30)}\,\bar{u}^3 + C^{(21)}\,\bar{u}^2\,\bar{v} + C^{(12)}\,\bar{u}\bar{v}^2 + C^{(03)}\,\bar{v}^3 + \cdots,
\end{aligned}
$$

where

$$C^{(20)} = \bar{b}'^2 - \tfrac{1}{2}\gamma^{(20)}, \qquad C^{(11)} = 2\bar{a}'\,\bar{b}' - \gamma^{(11)}, \qquad C^{(02)} = \bar{a}'^2 - \tfrac{1}{2}\gamma^{(02)},$$

$$C^{(30)} = -\bar{b}'^3 + \bar{b}'\,\gamma^{(20)} - \tfrac{1}{6}\gamma^{(30)},$$

$$C^{(21)} = -3\bar{a}'\,\bar{b}'^2 + \bar{a}'\,\gamma^{(20)} + 2\bar{b}'\,\gamma^{(11)} - \tfrac{1}{2}\gamma^{(21)}, \tag{19}$$

$$C^{(12)} = -3\bar{a}'^2\,\bar{b}' + \bar{b}'\,\gamma^{(02)} + 2\bar{a}'\,\gamma^{(11)} - \tfrac{1}{2}\gamma^{(12)},$$

$$C^{(03)} = -\bar{a}'^3 + \bar{a}'\,\gamma^{(02)} - \tfrac{1}{6}\gamma^{(03)}.$$

We therefore obtain for ξ and η the following expressions, the one for ξ being exact to terms of the *fourth* order, and the one for η being calculated only to terms of the third order:

$$\begin{aligned}\xi = {} & \bar{u} + (\tfrac{1}{2}\bar{a} - \bar{b}')\,\bar{u}^2 + \tfrac{1}{2}\bar{a}''\,\bar{v}^2 \\ & + A^{(30)}\,\bar{u}^3 + A^{(21)}\,\bar{u}^2\,\bar{v} + A^{(12)}\,\bar{u}\bar{v}^2 + A^{(03)}\,\bar{v}^3 \\ & + A^{(40)}\,\bar{u}^4 + A^{(31)}\,\bar{u}^3\,\bar{v} + A^{(22)}\,\bar{u}^2\,\bar{v}^2 + A^{(13)}\,\bar{u}\bar{v}^3 + A^{(04)}\,\bar{v}^4 + \cdots, \\ \eta = {} & \bar{v} + \tfrac{1}{2}\bar{b}\bar{u}^2 + (\tfrac{1}{2}\bar{b}'' - \bar{a}')\,\bar{v}^2 \\ & + B^{(30)}\,\bar{u}^3 + B^{(21)}\,\bar{u}^2\,\bar{v} + B^{(12)}\,\bar{u}\bar{v}^2 + B^{(03)}\,\bar{v}^3 + \cdots,\end{aligned} \tag{20}$$

where

$$A^{(30)} = \tfrac{1}{6}\alpha^{(30)} - \tfrac{1}{2}\bar{a}\bar{b}' + C^{(20)}, \qquad A^{(21)} = \tfrac{1}{2}\alpha^{(21)} - \bar{a}'\,\bar{b}' - \tfrac{1}{2}\bar{a}\bar{a}' + C^{(11)},$$

$$A^{(12)} = \tfrac{1}{2}\alpha^{(12)} - \tfrac{1}{2}\bar{a}''\,\bar{b}' - \bar{a}'^2 + C^{(02)}, \qquad A^{(03)} = \tfrac{1}{6}\alpha^{(03)} - \tfrac{1}{2}\bar{a}'\,\bar{a}'',$$

$$A^{(40)} = \tfrac{1}{24}\alpha^{(40)} - \tfrac{1}{6}\bar{b}'\,\alpha^{(30)} + \tfrac{1}{2}\bar{a}C^{(20)} + C^{(30)},$$

$$A^{(31)} = \tfrac{1}{6}\alpha^{(31)} - \tfrac{1}{2}\bar{b}'\,\alpha^{(21)} - \tfrac{1}{6}\bar{a}'\,\alpha^{(30)} + \bar{a}'\,C^{(20)} + \tfrac{1}{2}\bar{a}C^{(11)} + C^{(21)},$$

$$\begin{aligned}A^{(22)} = {} & \tfrac{1}{4}\alpha^{(22)} - \tfrac{1}{2}\bar{b}'\,\alpha^{(12)} - \tfrac{1}{2}\bar{a}'\,\alpha^{(21)} + \tfrac{1}{2}\bar{a}''\,C^{(20)} + \bar{a}'\,C^{(11)} \\ & + \tfrac{1}{2}\bar{a}C^{(02)} + C^{(12)},\end{aligned} \tag{21}$$

$$A^{(13)} = \tfrac{1}{6}\alpha^{(13)} - \tfrac{1}{6}\bar{b}'\,\alpha^{(03)} - \tfrac{1}{2}\bar{a}'\,\alpha^{(12)} + \tfrac{1}{2}\bar{a}''\,C^{(11)} + \bar{a}'\,C^{(02)} + C^{(03)},$$

$$A^{(04)} = \tfrac{1}{24}\alpha^{(04)} - \tfrac{1}{6}\bar{a}'\,\alpha^{(03)} + \tfrac{1}{2}\bar{a}''\,C^{(02)},$$

$$B^{(30)} = \tfrac{1}{6}\beta^{(30)} - \tfrac{1}{2}\bar{b}\bar{b}', \qquad B^{(21)} = \tfrac{1}{2}\beta^{(21)} - \bar{b}'^2 - \tfrac{1}{2}\bar{a}'\,\bar{b} + C^{(20)},$$

$$B^{(12)} = \tfrac{1}{2}\beta^{(12)} - \tfrac{1}{2}\bar{b}'\,\bar{b}'' - \bar{a}'\,\bar{b}' + C^{(11)},$$

$$B^{(03)} = \tfrac{1}{6}\beta^{(03)} - \tfrac{1}{2}\bar{a}'\,\bar{b}'' + C^{(02)}.$$

We wish to express ξ as a power series in $\bar{u}$ and η. We have from the second of (20), to terms of the fourth order,*

* Our subsequent developments are valid even though the last term in the second of equations (20) is of the third order.

$$\eta^4 = \bar{v}^4, \qquad \eta^3 = \bar{v}^3 + \tfrac{3}{2}\bar{b}\bar{u}^2\,\bar{v}^2 + \tfrac{3}{2}(\bar{b}'' - 2\bar{a}')\,\bar{v}^4,$$

$$\eta^2 = \bar{v}^2 + \bar{b}\bar{u}^2\,\bar{v} + (\bar{b}'' - 2\bar{a}')\,\bar{v}^3 + \tfrac{1}{4}\bar{b}^2\,\bar{u}^4 + 2B^{(30)}\,\bar{u}^3\,\bar{v} + [2B^{(21)} + \tfrac{1}{2}\bar{b}(\bar{b}'' - 2\bar{a}')]\,\bar{u}^2\,\bar{v}^2 + 2B^{(12)}\,\bar{u}\bar{v}^3 + [2B^{(03)} + \tfrac{1}{4}(\bar{b}'' - 2\bar{a}')^2]\,\bar{v}^4.$$

We have therefore, exact to terms of the fourth order,

$$\bar{v}^4 = \eta^4, \qquad \bar{u}\bar{v}^3 = \bar{u}\eta^3, \qquad \bar{u}^2\,\bar{v}^2 = \bar{u}^2\,\eta^2, \qquad \bar{u}^3\,\bar{v} = \bar{u}^3\,\eta,$$

$$\bar{v}^3 = \eta^3 - \tfrac{3}{2}\bar{b}\bar{u}^2\,\eta^2 - \tfrac{3}{2}(\bar{b}'' - 2\bar{a}')\,\eta^4,$$

$$\bar{u}\bar{v}^2 = \bar{u}\eta^2 - \bar{b}\bar{u}^3\,\eta - (\bar{b}'' - 2\bar{a}')\,\bar{u}\eta^3,$$

$$\bar{u}^2\,\bar{v} = \bar{u}^2\,\eta - \tfrac{1}{2}\bar{b}\bar{u}^4 - \tfrac{1}{2}(\bar{b}'' - 2\bar{a}')\,\bar{u}^2\,\eta^2,$$

$$\bar{v}^2 = \eta^2 - \bar{b}\bar{u}^2\,\eta - (\bar{b}'' - 2\bar{a}')\,\eta^3 + \tfrac{1}{4}\bar{b}^2\,\bar{u}^4 - 2B^{(30)}\,\bar{u}^3\,\eta + [\tfrac{3}{2}\bar{b}(\bar{b}'' - 2\bar{a}') - 2B^{(21)}]\,\bar{u}^2\,\eta^2 - 2B^{(12)}\,\bar{u}\eta^3 + [\tfrac{5}{4}(\bar{b}'' - 2\bar{a}')^2 - 2B^{(03)}]\,\eta^4.$$

Substituting these in the first of equations (20), we obtain

$$\begin{aligned}\xi = {} & \bar{u} + \tfrac{1}{2}(\bar{a} - 2\bar{b}')\,\bar{u}^2 + A^{(30)}\,\bar{u}^3 + [A^{(40)} - \tfrac{1}{2}\bar{b}A^{(21)} + \tfrac{1}{8}\bar{a}''\,\bar{b}^2]\,\bar{u}^4 \\ & + \cdots + \tfrac{1}{2}\bar{a}''\,\eta^2 + [A^{(03)} - \tfrac{1}{2}\bar{a}''(\bar{b}'' - 2\bar{a}')]\,\eta^3 \\ & + [A^{(04)} - \tfrac{3}{2}(\bar{b}'' - 2\bar{a}')\,A^{(03)} + \tfrac{5}{8}\bar{a}''(\bar{b}'' - 2\bar{a}')^2 - \bar{a}''\,B^{(03)}]\,\eta^4 \\ & + \cdots + \{(A^{(21)} - \tfrac{1}{2}\bar{a}''\,\bar{b})\,\bar{u}^2 + (A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)})\,\bar{u}^3 \\ & + \cdots\}\eta + \{A^{(12)}\,\bar{u} + [A^{(22)} - \tfrac{3}{2}\bar{b}A^{(03)} - \tfrac{1}{2}(\bar{b}'' - 2\bar{a}')\,A^{(21)} \\ & \qquad - \bar{a}''\,B^{(21)} + \tfrac{3}{4}\bar{a}''\,\bar{b}(\bar{b}'' - 2\bar{a}')]\,\bar{u}^2 + \cdots\}\eta^2 \\ & + \{[A^{(13)} - (\bar{b}'' - 2\bar{a}')\,A^{(12)} - \bar{a}''\,B^{(12)}]\,\bar{u} + \cdots\}\eta^3 \\ & + \cdots + \cdots.\end{aligned} \tag{22}$$

We have here arranged the expression for ξ in three parts. The first part is a power series in $\bar{u}$ alone, the second part is a power series in η alone, and the third part is a power series in η whose coefficients are power series in $\bar{u}$.

Since the triangle of reference is a covariant triangle, we might expect the coefficients in the development to be invariants; but not all of them are, because the parameter $\bar{u}$ is not chosen in any unique way. In fact, in the part containing powers of $\bar{u}$ alone, none of the coefficients is an invariant, while in the part containing powers of η alone all of the coefficients are invariants. In particular, the coefficient of η^3 vanishes identically:

$$A^{(03)} - \tfrac{1}{2}\bar{a}''(\bar{b}'' - 2\bar{a}') = 0. \tag{23}$$

Again, in the part which contains powers of both $\bar{u}$ and η, we have enclosed the coefficient of each power of η in a brace $\{\ \}$. Only the first coefficient in each brace is an invariant. Thus, in the term

$$(24)\qquad \{(A^{(21)} - \tfrac{1}{2}\bar{a}''\,\bar{b})\,\bar{u}^2 + (A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)})\,\bar{u}^3 + \cdots\}\eta,$$

the quantity $A^{(21)} - \frac{1}{2}\bar{a}''\,\bar{b}$ is an invariant, while $A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)}$ is not. In fact, following out the notation of Wilczynski, we may verify that

$$A^{(21)} - \tfrac{1}{2}\bar{a}''\,\bar{b} = -\tfrac{1}{2}(\bar{H} + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}),$$

$$(25)\qquad A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)} = -\tfrac{1}{2}\bar{\mathfrak{B}}(\bar{\mathfrak{A}}'', \bar{B}') - \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}\bar{\mathfrak{B}}' - \tfrac{1}{6}(\bar{H}, \bar{B}') - \tfrac{1}{2}(\bar{a} - 2\bar{b}')\,(\bar{H} + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}),$$

where

$$\bar{H} = \bar{c}' + \bar{a}'\,\bar{b}' - \bar{a}_u', \qquad \bar{\mathfrak{A}}'' = \bar{a}'', \qquad \bar{\mathfrak{B}} = \bar{b}, \qquad \bar{\mathfrak{B}}' = \tfrac{1}{3}(2\bar{b}' - \bar{a}) + \frac{1}{6}\frac{\bar{b}_u}{\bar{b}},$$

$$(\bar{\mathfrak{A}}'', \bar{B}') = \bar{\mathfrak{A}}_u'' - \bar{B}'\,\bar{\mathfrak{A}}'' = \bar{a}_u'' + (\bar{a} - 2\bar{b}')\,\bar{a}'',$$

$$(\bar{H}, \bar{B}') = \bar{H}_u + \bar{B}'\,\bar{H} = \bar{H}_u - (\bar{a} - 2\bar{b}')\,\bar{H}$$

are invariants of the canonical net. Then the second of (25) is not invariant, since the quantity $\bar{a} - 2\bar{b}'$ is not.

If instead of the parameter $\bar{u}$ we choose a uniquely determined parameter τ, it will be possible to transform (22) into a new expansion in which all the coefficients are relative invariants. This may be done by taking for τ the part of (22) containing powers of $\bar{u}$ alone, i. e.,

$$(26)\qquad \tau = \bar{u} + \tfrac{1}{2}(\bar{a} - 2\bar{b}')\,\bar{u}^2 + A^{(30)}\,\bar{u}^3 + \cdots,$$

the inversion of which gives

$$\bar{u} = \tau - \tfrac{1}{2}(\bar{a} - 2\bar{b}')\,\tau^2 + \cdots,$$

$$\bar{u}^2 = \tau^2 - (\bar{a} - 2\bar{b}')\,\tau^3 + \cdots.$$

Each brace in (22) becomes a new power series in τ, with invariants as coefficients; thus, in particular, the term (24) becomes

$$\{-\tfrac{1}{2}(\bar{H} + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}})\,\tau^2 - [\tfrac{1}{2}\bar{\mathfrak{B}}(\bar{\mathfrak{A}}'', \bar{B}') + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}\bar{\mathfrak{B}}' - \tfrac{1}{6}(\bar{H}, \bar{B}')]\,\tau^3 + \cdots\}\eta.$$

The coefficient of η^4 in (22) is found without much difficulty to have the value

$$-\frac{\bar{\mathfrak{A}}''}{8}(\bar{\mathfrak{A}}'', \bar{B}'),$$

so that we have the development

$$
\begin{aligned}
\xi = \tau &+ \tfrac{1}{2}\bar{a}''\,\eta^2 - \tfrac{1}{8}\bar{a}''(\bar{\mathfrak{A}}'', \bar{B}')\,\eta^4 + \cdots \\
&+ \{\theta^{(21)}\tau^2 + \theta^{(31)}\tau^3 + \cdots\}\eta \\
&+ \{\theta^{(12)}\tau + \theta^{(22)}\tau^2 + \cdots\}\eta^2 \\
&+ \{\theta^{(13)}\tau + \cdots\}\eta^3 + \cdots,
\end{aligned} \tag{27}
$$

where $\theta^{(ij)}$, the coefficient of $\tau^i\,\eta^j$, is a relative invariant of the canonical net.

We may make some further transformations, which will yield a development with absolute invariants as coefficients. The coefficients in (26) are of course constants, being the values of the corresponding invariants for the values 0, 0 of the arguments $\bar{u}$, $\bar{v}$. Consider the invariant $\theta^{(ij)}$; it is easily verified that the point transformation

$$\bar{u} = U(\bar{u}), \qquad \bar{v} = V(\bar{v})$$

changes it into

$$\bar{\theta}^{(ij)} = \frac{1}{(U')^{i-1}} \cdot \frac{1}{(V')^{j}}\theta^{(ij)}, \tag{28}$$

where U' and V' are the first derivatives of U and V. The same transformation changes the invariants $\bar{a}''$ and $\bar{b}$ into

$$\bar{a}'' = \frac{U'}{V'^2}\bar{a}'', \qquad \bar{b} = \frac{V'}{U'^2}\bar{b}.$$

Therefore

$$\bar{a}''\,\bar{b}^2 = \frac{1}{U'^3}\bar{a}''\,\bar{b}^2, \qquad \bar{a}''^2\,\bar{b} = \frac{1}{V'^3}\bar{a}''^2\,\bar{b}. \tag{29}$$

Let us put

$$\mu = \sqrt[3]{\bar{a}''\,\bar{b}^2}, \qquad \nu = \sqrt[3]{\bar{a}''^2\,\bar{b}}. \tag{30}$$

Then the quantity

$$I^{(ij)} = \mu \cdot \left(\frac{1}{\mu}\right)^i \left(\frac{1}{\nu}\right)^j \theta^{(ij)}$$

is an *absolute invariant.* In fact, the transformed quantity is seen from (27) and (28) to satisfy the equation

$$\bar{I}^{(ij)} = I^{(ij)}.$$

If, then, in the expansion (26) we put*

$$x = \mu\xi, \qquad y = \nu\eta, \qquad t = \mu\tau, \tag{31}$$

we obtain the final canonical development,

$$
\begin{aligned}
x = t &+ \tfrac{1}{2}y^2 + I^{(04)}\,y^4 + \cdots \\
&+ \{I^{(21)}\,t^2 + I^{(31)}\,t^3 + \cdots\}y \\
&+ \{I^{(12)}\,t + I^{(22)}\,t^2 + \cdots\}y^2 \\
&+ \{I^{(13)}\,t + \cdots\}y^3 + \cdots,
\end{aligned} \tag{32}
$$

the coefficients of which are all absolute invariants.

* The transformation (31) leaves fixed the covariant triangle $P_{\bar{y}}$, $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$.

In recapitulation, we state the result:

A one-parameter family of analytic plane curves, given by the equations

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3)$$

for u = const., may be represented in the neighborhood of a regular point by the canonical development (31) *for t = const. In this development all the coefficients are absolute projective invariants. The triangle of reference which gives rise to this development is the covariant triangle $P_{\bar{y}}$, $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$, of which a purely geometric interpretation has been given. The parameter t is uniquely determined as an analytic function of the old parameter u.*

In the derivation of this canonical development, the essential steps to be noted are first, the determination of a triangle of reference which bears a covariant relation to the one-parameter family of curves (considered apart from any net of which it may be a component family), and secondly the characteristic choice of the parameter t. From the results stated in § 1, we see that the absolute invariants which enter into the canonical development as coefficients are all expressible rationally in terms of the coefficients of the original differential equations which define the family of curves as a component of a net in which the second family bears only an accidental relation to the given family. Our canonical development is therefore independent of the integration which is necessary to determine the canonical net.

COLLEGE OF THE CITY OF NEW YORK,
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THE MINIMUM OF A DEFINITE INTEGRAL FOR UNILATERAL VARIATIONS IN SPACE*

BY

G. A. BLISS AND A. L. UNDERHILL

For the problem of the calculus of variations with fixed end points, the curves which minimize an integral of the form

$$J = \int F(x, y, z, x', y', z')\, ds$$

have previously been studied, provided that the minimizing arc along which the integral J is taken is free to traverse any portion of space or restricted to lie entirely within a given region R. It may happen, however, that while no extremal joining the two fixed end points exists, which is entirely interior to a region R, yet there may be a minimizing curve consisting partly of arcs interior to the region and partly of arcs along its boundary. This gives rise to the problem of investigating the properties which characterize a minimizing

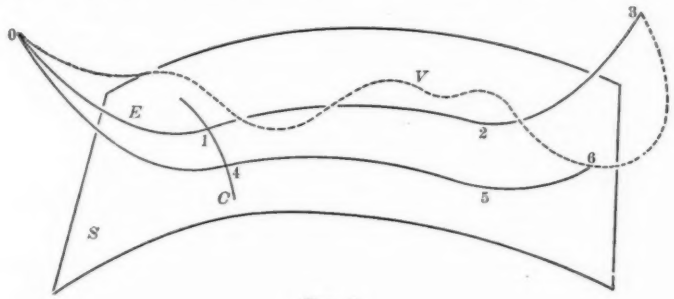

FIG. 1.

curve lying partly on a given surface S, while the variations with which it is compared are restricted to lie on one side of the surface, a problem analogous to one in the plane which has been discussed by Bliss.† For this problem in space a number of necessary conditions have been derived by other writers, but the theory of the analogue of Jacobi's condition is still incomplete.

* Presented to the Society, December 27, 1913.

† These Transactions, vol. 5 (1904), pp. 477–492.

Suppose that the arc E, whose minimizing properties are to be investigated, consists of three parts E_{01}, E_{12}, and E_{23}, as shown in the accompanying figure (Fig. 1). The arc E_{12} must be an extremal of the integral J on the surface S, while the other two, lying on the same side of S and joining the end points of E_{12} to the two fixed points 0 and 3, are necessarily extremals in space and tangent to the arc E_{12}. For the corresponding situation in the plane, the analogue of Jacobi's condition requires only that Jacobi's condition itself be satisfied by each of the arcs E_{01} and E_{23}. For the problem here considered, however, there is in general more required than this. It is not sufficient, in order that E shall minimize the integral, that no conjugate point to 0 lies on the extremal arc E_{01}, and no conjugate point to 2 on the arc E_{23}.*

It will be shown in the following pages that from the two parameter family of space extremals passing through the point 0, a set containing E_{01} and depending upon a single parameter may be selected, each curve of which is tangent to the surface S at a point of a curve C through the point 1. Through the points of C a one parameter family of extremals on S can be passed, each one of which is tangent at a point of C to one of the space extremals through the point 0. Again at any point of one of the single infinitude of surface extremals so determined, there is a space extremal tangent to the surface extremal and lying on the required side of the surface. These form a two parameter family of space extremals tangent to S.

Two new conditions, analogues of the Jacobi condition in the plane case, will be established; (1) the arc E_{12} must not contain a point of contact with the enveloping curve of the one-parameter family of surface extremals above mentioned; and (2) the arc E_{23} must not contain a contact point with the enveloping surface of the two parameter family of space extremals constructed as above tangent to the surface S.

The necessary conditions hitherto derived and the two analogues to the Jacobi condition described in the preceeding paragraph, when strengthened in a manner customary in the calculus of variations, are sufficient to insure that the curve E actually minimizes the integral J.

1. Necessary Conditions Hitherto Derived

Let the curve E joining the points 0 and 3, whose minimizing properties are to be studied, have the equations

$$(E) \qquad x = x(s), \quad y = y(s), \quad z = z(s) \qquad (s_0 \leqq s \leqq s_1),$$

where s is the length of arc measured from the point 0. The curve is supposed

* See a remark by Hadamard in his *Leçons sur le calcul des variations*, p. 457.

not to intersect itself and is made up of three pieces, each of class C'',* as shown in Fig. 1.

The surface S, on which E_{12} lies, has equations of the form

$$(S) \qquad x = \xi(\alpha, \beta), \qquad y = \eta(\alpha, \beta), \qquad z = \zeta(\alpha, \beta).$$

It has no points in common with E_{01} and E_{23} except 1 and 2, and is further supposed to have no multiple or singular points and to be of class C''' in the vicinity of E_{12}.

The function F in the integral J is assumed to have the following properties in a region R of points (x, y, z, x', y', z') for which (x, y, z) ranges over a neighborhood of E, while (x', y', z') are any values not all zero:

$$(A) \qquad F \text{ is of class } C''';$$

$$(B) \qquad F(x, y, z, \kappa x', \kappa y', \kappa z') = \kappa F(x, y, z, x', y', z') \qquad (\kappa > 0);$$

$$(C) \qquad F_1 \neq 0.$$

The property (B) is the usual homogeneity property of the function F which makes the value of the integral independent of the parametric representation of the curve along which it is taken, while (C) specifies that the problem is a regular one.†

If E is to minimize J, then it must satisfy a number of necessary conditions which are already known and which may be stated as follows:

(I) The arcs E_{01} and E_{23} must be space extremals satisfying the differential equations

$$(1) \qquad J^{(x)} = F_x - \frac{d}{ds}F_{x'} = 0, \qquad J^{(y)} = F_y - \frac{d}{ds}F_{y'} = 0, \qquad J^{(z)} = F_z - \frac{d}{ds}F_{z'} = 0;$$

(II) Along the arc E_{12} the equations

$$(2) \qquad \frac{J^{(x)}}{X} = \frac{J^{(y)}}{Y} = \frac{J^{(z)}}{Z} = \lambda(s)$$

must be satisfied, where X, Y, Z are the direction cosines of the normal of the surface S in the direction toward the side of S on which E_{01} and E_{23} lie, while $\lambda(s)$ is merely the notation for the function of s defined by the three ratios;

* An arc is said to be of class $C^{(n)}$ if the derivatives of the functions defining it up to and including those of order n are continuous, and if the expression $\sqrt{x'^2 + y'^2 + z'^2}$ is everywhere different from zero along it.

† For the definition of F_1 and a "regular problem" see a paper by Mason and Bliss, these Transactions, vol. 9 (1908), p. 444. This paper will be referred to hereafter as M. and B.

(III) Along E_{12} the inequality $\lambda(s) \gtreqless 0$ must hold;

(IV) The function

$$E(x, y, z; x', y', z'; p, q, r) = F(x, y, z, p, q, r) - pF_{x'}(x, y, z, x', y', z') - qF_{y'}(x, y, z, x', y', z') - rF_{z'}(x, y, z, x', y', z')$$

must be positive or zero for all values (x, y, z, x', y', z') on the arc E_{03} and for arbitrary values $(p, q, r) \neq (0, 0, 0)$;

(V) At the points 1 and 2, E_{01} and E_{23} must be tangent to the arc E_{12};

(VI) The arc E_{01} must not contain a point conjugate to 0 between 0 and 1.*

From the property IV it follows that the quadratic form

$$Q(x, y, z; x', y', z'; p, q, r) = F_{x'x'}p^2 + F_{y'y'}q^2 + F_{z'z'}r^2 + 2F_{y'z'}qr + 2F_{z'x'}rp + 2F_{x'y'}pq$$

must be positive or zero for the arguments described for E.† The sign of F_1 must therefore be positive, and it can be shown that $Q \geqq 0$ for all values (x, y, z, x', y', z') in the region R and for arbitrary directions (p, q, r).‡ Furthermore Q vanishes only when the directions x', y', z' and p, q, r are in the same line.§ The same properties hold for the E-function, with the further restriction that E vanishes only when x', y', z' and p, q, r are identical (not opposite) directions.‖

The condition II can also be stated in terms of the integrand

$$f(\alpha, \beta, \alpha', \beta') = F(\xi, \eta, \zeta, \xi_\alpha \alpha' + \xi_\beta \beta', \eta_\alpha \alpha' + \eta_\beta \beta', \zeta_\alpha \alpha' + \zeta_\beta \beta')$$

of the integral

$$J = \int f(\alpha, \beta, \alpha', \beta')\, ds \tag{3}$$

* For the conditions (I), (IV), (VI) on the arcs E_{01} and E_{23}, see M. and B., loc. cit., pages 443, 455, 453, respectively. For (II), (III), (V), see Hadamard, loc. cit., pages 133, 179, 456. There is an exceptional case to (VI) mentioned in M. and B., p. 453, which can, however, be treated by a consideration of the second variation instead of the geometric proof used by them. The proof that the condition (IV) must hold along the arc E_{12} can be made in a manner analogous to that used for the other arcs. It is somewhat more difficult because the variations are restricted to lie on one side of the surface S.

† M. and B., p. 455.

‡ The argument is like that of M. and B., p. 456, for the functions $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$.

§ M. and B., p. 456, formulas (40).

‖ The proof of this statement may be made by means of the formulas in M. and B., but more elegantly by the use of the results of Behaghel, Mathematische Annalen, vol. 73 (1913), p. 596. The functions A, B, C used by him may become infinite for directions parallel to the axes and the validity of his proof is questionable in such cases, though his result is correct. The justification for these statements will appear in the near future in a paper by Bliss.

taken along curves on the surface S. Along E_{12} the parameters α and β are functions

$$\alpha = \alpha(s), \qquad \beta = \beta(s) \qquad (s_1 \leqq s \leqq s_2)$$

which are of class C''. After an easy calculation the Euler equations for the integral (3) are seen to have the form

$$\begin{aligned} f_\alpha - \frac{d}{ds} f_{\alpha'} &= J^{(x)}\,\xi_\alpha + J^{(y)}\,\eta_\alpha + J^{(z)}\,\zeta_\alpha = 0, \\ f_\beta - \frac{d}{ds} f_{\beta'} &= J^{(x)}\,\xi_\beta + J^{(y)}\,\eta_\beta + J^{(z)}\,\zeta_\beta = 0, \end{aligned} \tag{4}$$

and these are satisfied along E_{12} on account of the condition II.

Furthermore the function*

$$f_1 = \frac{f_{\alpha'\alpha'}}{\beta'^2} = -\frac{f_{\alpha'\beta'}}{\alpha'\beta'} = \frac{f_{\beta'\beta'}}{\alpha'^2}$$

has the values

$$f_1 = \frac{Q(x, y, z; x', y', z'; \xi_\alpha, \eta_\alpha, \zeta_\alpha)}{\beta'^2} = \frac{Q(x, y, z; x', y', z'; \xi_\beta, \eta_\beta, \zeta_\beta)}{\alpha'^2},$$

where

$$x' = \xi_\alpha\,\alpha' + \xi_\beta\,\beta', \qquad y' = \eta_\alpha\,\alpha' + \eta_\beta\,\beta', \qquad z' = \zeta_\alpha\,\alpha' + \zeta_\beta\,\beta'.$$

One of these expressions is always well defined and different from zero, since the derivatives x', y', z' can coincide with at most one of the coördinate directions on the surface. The problem of minimizing the integral (3) on the surface is therefore a regular one, and E_{12} is one of its extremals.

If a space extremal satisfying the condition I is tangent to a surface extremal satisfying II, then at the point of contact there is a relation between their curvatures which will be of service later. Let the elements of the second order on the two extremals at the point of contact be, respectively,

$$(x, y, z, x', y', z', \bar{x}'', \bar{y}'', \bar{z}'')$$

and

$$(x, y, z, x', y', z', x'', y'', z'').$$

From equations (1) and (2) it follows that

$$\lambda X = J^{(x)} - \bar{J}^{(x)}, \qquad \lambda Y = J^{(y)} - \bar{J}^{(y)}, \qquad \lambda Z = J^{(z)} - \bar{J}^{(z)}, \tag{5}$$

and

$$\begin{aligned} \lambda\{X(x'' - \bar{x}'') + Y(y'' - \bar{y}'') + Z(z'' - \bar{z}'')\} \\ &= (x'' - \bar{x}'')\,(J^{(x)} - \bar{J}^{(x)}) + (y'' - \bar{y}'')\,(J^{(y)} - \bar{J}^{(y)}) \\ &\qquad + (z'' - \bar{z}'')\,(J^{(z)} - \bar{J}^{(z)}) \\ &= Q(x, y, z; x', y', z'; x'' - \bar{x}'', y'' - \bar{y}'', z'' - \bar{z}''). \end{aligned}$$

* See Bolza, *Vorlesungen über Variationsrechnung*, p. 196.

This last expression can vanish, however, only when

$$x'' - \bar{x}'' = \kappa x', \qquad y'' - \bar{y}'' = \kappa y', \qquad z'' - \bar{z}'' = \kappa z',$$

where κ is a value which may be zero. In case it does vanish the equations (5) show that $\lambda = 0$, as a result of the homogeneity properties of the function F.* Hence the following lemma is true:

LEMMA 1: *If $\lambda \neq 0$ at the point of contact of a surface extremal with a space extremal, then at that point the inequality*

$$Xx'' + Yy'' + Zz'' \neq X\bar{x}'' + Y\bar{y}'' + Z\bar{z}''$$

holds; or, in other words, the projections of the first curvatures, $1/r$ and $1/\bar{r}$, of the two extremals on the normal to the surface are different.

From this point in the paper it will be assumed that the arc E_{03} satisfies the conditions I, II, IV, V and the modified conditions

(III′) $\lambda(s) > 0$ along E_{12};

(VI′) the arc E_{01} does not contain a point conjugate to 0 between 0 and 1 or at 1.†

On the basis of these assumptions further conditions, analogous to Jacobi's conditions for the usual form of the problem, will be derived.

2. The Three Families of Extremals Associated with E_{03}

The totality of solutions of the differential equations (1) passing through 0 contains two arbitrary parameters. As a result of the hypotheses made upon the function F they may be represented in the form

$$(6)\quad x = \phi(s, u, v, w), \qquad y = \psi(s, u, v, w), \qquad z = \chi(s, u, v, w),$$

$$u^2 + v^2 + w^2 = 1,$$

the parameters u, v, w being the direction cosines of the positive tangent to the extremal at the point 0. The particular extremal E_{01} is contained in this set for $u = u_0, v = v_0, w = w_0$, and the functions $\phi, \psi, \chi, \phi_s, \psi_s, \chi_s$ are of class C' in neighborhoods of the two sets of parameter values

$$s = 0, \qquad u^2 + v^2 + w^2 = 1;$$

$$0 \leqq s \leqq s_1, \qquad u = u_0, \qquad v = v_0, \qquad w = w_0$$

of which the second set defines the curve E_{01}. At the point 0, where $s = 0$, the following identities in u, v, w hold

$$x_0 = \phi(0, u, v, w), \qquad y_0 = \psi(0, u, v, w), \qquad z_0 = \chi(0, u, v, w).$$

* M. and B., p. 441, formula (5).

† The condition VI′ is necessary except in one singular case. See M. and B., p. 453.

Let the notation $\Delta(s, u, v, w)$ designate the determinant

$$\Delta(s, u, v, w) = \begin{vmatrix} \phi_s & \phi_u & \phi_v & \phi_w \\ \psi_s & \psi_u & \psi_v & \psi_w \\ \chi_s & \chi_u & \chi_v & \chi_w \\ 0 & u & v & w \end{vmatrix}.$$

On account of the condition VI′ it follows* that

$$\Delta(s, u_0, v_0, w_0) \neq 0 \qquad (0 < s \leqq s_1). \tag{7}$$

The equations

$$\begin{aligned} \xi(\alpha, \beta) + \omega X &= \phi(s, u, v, w), \\ \eta(\alpha, \beta) + \omega Y &= \psi(s, u, v, w), \\ \zeta(\alpha, \beta) + \omega Z &= \chi(s, u, v, w), \end{aligned}$$

have an initial solution $(\alpha, \beta, \omega; s, u, v, w) = (\alpha_1, \beta_1, 0; s_1, u_0, v_0, w_0)$ corresponding to the point 1, at which the functional determinant for α, β, ω is different from zero. Hence these equations have solutions

$$\alpha = \alpha(s, u, v, w), \qquad \beta = \beta(s, u, v, w), \qquad \omega = \omega(s, u, v, w) \tag{8}$$

of class C'' near (s_1, u_0, v_0, w_0) and reducing to $(\alpha_1, \beta_1, 0)$ at these values. The derivatives of ω at the point 1 are readily calculated to be

$$\begin{aligned} &\omega_s = X\phi_s + Y\psi_s + Z\chi_s, \qquad \omega_u = X\phi_u + Y\psi_u + Z\chi_u, \\ &\omega_v = X\phi_v + Y\psi_v + Z\chi_v, \qquad \omega_w = X\phi_w + Y\psi_w + Z\chi_w, \\ &\omega_{ss} = X\phi_{ss} + Y\psi_{ss} + Z\chi_{ss} - L\alpha_s^2 - 2M\alpha_s\beta_s - N\beta_s^2\dagger \\ &\qquad = Xx'' + Yy'' + Zz'' - (X\bar{x}'' + Y\bar{y}'' + Z\bar{z}''), \end{aligned} \tag{9}$$

where the notations in the last line are the same as those of the preceding section. The set (x'', y'', z'') belongs to E_{01} at the point 1, and the set $(\bar{x}'', \bar{y}'', \bar{z}'')$ to E_{12}.

The solutions (s, u, v, w) of the equations

$$\omega(s, u, v, w) = 0, \qquad \omega_s(s, u, v, w) = 0, \qquad u^2 + v^2 + w^2 = 1 \tag{10}$$

determine points where the extremal (6) is tangent to the surface S. On account of the tangency of E_{01} and E_{12}, it is evident that among the solutions are the values (s_1, u_0, v_0, w_0). On the other hand the determinants of the matrix

$$\left\| \begin{matrix} \omega_{ss} & \omega_{su} & \omega_{sv} & \omega_{sw} \\ \omega_s & \omega_u & \omega_v & \omega_w \\ 0 & u & v & w \end{matrix} \right\|$$

* See M. and B., §3 and p. 453.

† For the definitions of L, M, N and their relations to the first curvature, see Scheffers, *Anwendung der Differential und Integral-Rechnung auf Geometrie*, vol. 2, page 494 (*D*).

cannot all vanish at (s_1, u_0, v_0, w_0). At these values $\omega_s = 0$ since E_{01} is tangent to the surface S; but from equations (9) and the property (7) it follows readily that the determinants of the last two rows can not all vanish; and from III′ and the lemma of the preceding section, ω_{ss} is different from zero. The equations (10) therefore have solutions for s and two of the variables u, v, w in terms of the third, passing through the values (s_1, u_0, v_0, w_0) and of class C' near them.

These solutions and the equations (8) *determine a one parameter family of values*

$$\alpha = \alpha(\tau), \qquad \beta = \beta(\tau), \qquad s = \sigma(\tau),$$
$$u = u(\tau), \qquad v = v(\tau), \qquad w = w(\tau)\,*$$

of class C' near $\tau = 0$, which in turn determine a curve

$$(C) \qquad \alpha = \alpha(\tau), \qquad \beta = \beta(\tau)$$

on the surface S and a one parameter family

$$(11) \qquad \begin{aligned} x &= \phi[s, u(\tau), v(\tau), w(\tau)] = \phi(s, \tau), \\ y &= \psi[s, u(\tau), v(\tau), w(\tau)] = \psi(s, \tau), \\ z &= \chi[s, u(\tau), v(\tau), w(\tau)] = \chi(s, \tau), \end{aligned}$$

of the extremals (6). *Through each point of C there passes a unique extremal of the set tangent to the surface S. The functions $\phi(s, \tau)$ and $\phi_s(s, \tau)$ are of class C' near the values $s_0 \leqq s \leqq s_1$, $\tau = 0$, which define the arc E_{01}, with similar properties for ψ and χ.*

On account of the hypotheses on the function F and the resulting properties of f, it is possible to prove the existence of a one parameter family of extremals on the surface S satisfying the differential equations II or (4), and such that each extremal passes through a point of C, and has the direction determined by the corresponding space extremal of the set (11) tangent to S at that point. Let the length of arc along the surface extremal be denoted by σ and be measured so that $\sigma - \sigma(\tau)$ is the length of arc from C.

There exists, then, a one parameter family of surface extremals with equations on the surface of the form

$$(12) \qquad \alpha = A(\sigma, \tau), \qquad \beta = B(\sigma, \tau),$$

or, in space, of the form

$$(13) \qquad \begin{aligned} x &= \xi[A(\sigma, \tau), \ B(\sigma, \tau)] = \xi(\sigma, \tau), \\ y &= \eta[A(\sigma, \tau), \ B(\sigma, \tau)] = \eta(\sigma, \tau), \\ z &= \zeta[A(\sigma, \tau), \ B(\sigma, \tau)] = \zeta(\sigma, \tau), \end{aligned}$$

* The parameter τ will be $u - u_0$, $v - v_0$ or $w - w_0$.

where the functions $\xi(\sigma, \tau)$ *and* $\xi_\sigma(\sigma, \tau)$ *are of class* C' *near the values*

$$s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0,$$

defining the arc E_{12}. *Along the curve* C *the relations*

$$\phi[\sigma(\tau), \tau] = \xi[\sigma(\tau), \tau], \qquad \phi_s[\sigma(\tau), \tau] = \xi_\sigma[\sigma(\tau), \tau],$$

with similar ones for ψ *and* χ, *are identities in* τ. *They express the fact that at a point of* C *the surface extremal* (12) *and the space extremal* (11) *are tangent to each other.*

Through each point of an extremal on the surface S determined by the equations (12), there passes a space extremal whose direction at the point of intersection coincides with that of the extremal on the surface. This extremal must satisfy the differential equations I. If its length of arc s is measured from the initial value σ at the point of contact with the surface, the existence theorems for differential equations and the properties of the function F justify the following theorem:

THEOREM. *There exists a two parameter family of space extremals tangent to the extremals* (13) *on the surface* S, *and having equations of the form*

$$(14) \qquad x = \Phi(s, \sigma, \tau), \qquad y = \Psi(s, \sigma, \tau), \qquad z = \mathrm{X}(s, \sigma, \tau),$$

where Φ *and* Φ_s *are of class* C' *near the values*

$$s = \sigma, \qquad s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0,$$

$$s_2 \leqq s \leqq s_3, \qquad \sigma = s_2, \qquad \tau = 0,$$

defining, respectively, the arcs E_{12} *and* E_{23}. *Similar properties hold for* Ψ *and* X. *The identities*

$$(15) \qquad \Phi(\sigma, \sigma, \tau) \equiv \xi(\sigma, \tau), \qquad \Phi_s(\sigma, \sigma, \tau) \equiv \xi_\sigma(\sigma, \tau),$$

with similar ones for the other functions, express the tangency of (14) *with the surface extremal* (13) *at their point of intersection.*

The identities (15) give rise to a number of further relations important in the following pages. By differentiating the first one and applying the second, it follows that

$$\Phi_\sigma(\sigma, \sigma, \tau) \equiv 0, \qquad \Phi_\tau(\sigma, \sigma, \tau) = \xi_\tau(\sigma, \tau),$$

and by differentiating the second,

$$\Phi_{s\sigma}(\sigma, \sigma, \tau) \equiv \xi_{\sigma\sigma}(\sigma, \tau) - \Phi_{ss}(\sigma, \sigma, \tau) \equiv \bar{x}'' - x''$$

with similar relations for Ψ and X. The determinant

$$\Delta(s, \sigma, \tau) = \begin{vmatrix} \Phi_s & \Phi_\sigma & \Phi_\tau \\ \Psi_s & \Psi_\sigma & \Psi_\tau \\ \mathrm{X}_s & \mathrm{X}_\sigma & \mathrm{X}_\tau \end{vmatrix}$$

has therefore the derivative

$$\Delta_s(s,\sigma,\tau)=\begin{vmatrix}\Phi_s & \Phi_{s\sigma} & \Phi_\tau\\ \Psi_s & \Psi_{s\sigma} & \Psi_\tau\\ \mathrm{X}_s & \mathrm{X}_{s\sigma} & \mathrm{X}_\tau\end{vmatrix}=\begin{vmatrix}x''-\bar{x}'' & \xi_\sigma & \xi_\tau\\ y''-\bar{y}'' & \eta_\sigma & \eta_\tau\\ z''-\bar{z}'' & \zeta_\sigma & \zeta_\tau\end{vmatrix} \tag{16}$$

for the values (σ,σ,τ) along the surface S. But at a point where the functional determinant of the functions A and B is different from zero, the relations (13) show that this has the value

$$\Delta_s(\sigma,\sigma,\tau)=K\{X(x''-\bar{x}'')+Y(y''-\bar{y}'')+Z(z''-\bar{z}'')\}$$

where K is a factor of proportionality which does not vanish. According to the lemma of § 1, the following result may be stated:

LEMMA 2. *The derivative $\Delta_s(s,\sigma,\tau)$ is different from zero at any point of the curve E_{12} on the surface S at which the determinant*

$$\begin{vmatrix}A_\sigma & A_\tau\\ B_\sigma & B_\tau\end{vmatrix}$$

is different from zero.

3. Auxiliary Theorems

The computations of the following pages can be considerably simplified by the use of some auxiliary theorems, the first of which may be stated as follows:

THEOREM. *If a family of curves*

$$x=\phi(s,\lambda),\qquad y=\psi(s,\lambda),\qquad z=\chi(s,\lambda) \tag{17}$$

satisfies the relation

$$J^{(x)}\phi_\lambda+J^{(y)}\psi_\lambda+J^{(z)}\chi_\lambda\equiv 0,$$

then the derivative of the integral

$$\int_{s_0}^{s_1}F(\phi,\psi,\chi,\phi_s,\psi_s,\chi_s)\,ds,$$

when s_0, s_1, and λ are functions $s_0(v)$, $s_1(v)$, $\lambda(v)$, is

$$[F_{x'}x_v+F_{y'}y_v+F_{z'}z_v]_0^1.$$

The arguments of the derivatives of F are the same as those in the integral, and

$$x_v=\phi_s s_v+\phi_\lambda\lambda_v,\qquad y_v=\psi_s s_v+\psi_\lambda\lambda_v,\qquad z_v=\chi_s s_v+\chi_\lambda\lambda_v. \tag{18}$$

This theorem follows at once by known methods* provided that the func-

* See M. and B., §2.

tions s_0, s_1, λ are of class C' for $|v| \leqq \epsilon$, and ϕ, ϕ_s of class C' in the region

$$s_0(v) \leqq s \leqq s_1(v), \qquad \lambda = \lambda(v) \qquad (|v| \leqq \epsilon),$$

with similar properties for ψ and χ.

THEOREM. *If the curves of a second family*

$$x = \Phi(t, \mu), \qquad y = \Psi(t, \mu), \qquad z = \mathrm{X}(t, \mu) \tag{19}$$

adjoin those of the first family for parameter values

$$s = s_1(v), \qquad \lambda = \lambda(v); \qquad t = t_1(v), \qquad \mu = \mu(v),$$

and if corresponding arcs of the two families have the same direction at their common point, then the function

$$G(v) = \int_{s_0}^{s_1} F(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s)\, ds + \int_{t_1}^{t_2} F(\Phi, \Psi, \mathrm{X}, \Phi_t, \Psi_t, \mathrm{X}_t)\, dt, \tag{20}$$

in which t_2 is a function of v as well as t_1 and μ, has the derivative

$$G'(v) = [\, F_{x'} x_v + F_{y'} y_v + F_{z'} z_v \,]_0^2. \tag{21}$$

The expressions in the bracket must be formed at the point 0 for the first family, and at the point 2 for the second, as described in the first theorem.

The derivative of $G(v)$ has the value

$$[\, F_{x'} x_v + F_{y'} y_v + F_{z'} z_v \,]_0^1 + [\, F_{x'} x_v + F_{y'} y_v + F_{z'} z_v \,]_1^2,$$

from the first theorem. At the point 1 the two brackets cancel on account of the homogeneity properties of F and the relations

$$\phi[s_1(v), \lambda(v)] = \Phi[t_1(v), \mu(v)],$$

$$\phi_s[s_1(v), \lambda(v)] = \kappa(v)\, \Phi_t[t_1(v), \mu(v)] \qquad [\kappa(v) > 0],$$

with similar ones for the other functions, which express the intersection and the tangency of the two families. It is understood that the continuity properties of the two families are like those described above for the first.

A similar theorem evidently holds when the number of families is more than two. If the curves of the first family all pass through a fixed point 0 for $s = s_0(v)$, $\lambda = \lambda(v)$, the bracket in (21) vanishes at 0 since the same is true of the derivatives of (17). If the curves of the second family similarly pass through a fixed point 2 for $t = t_2(v)$, $\mu = \mu(v)$, then $G(v)$ is a constant.

The family (19) may be independent of μ, in which case it will be simply an enveloping curve of the family (17). If in this case $t_2(v)$ is a constant, the second of the integrals (20) will be simply the value of J taken along the envelope from 1 to a fixed point 2, and the function $G(v)$ will again be a constant.

4. Jacobi's Conditions for Unilateral Variations

As a result of the assumptions made on the curve E_{03} at the end of §1, it is possible to prove the following further necessary conditions for a minimum:

(VII) *The curve E_{03} must not have a point of contact between* 1 *and* 2 *with an enveloping curve d of the family* (13) *of extremals on the surface S.*

The proof will be made here for the case where the enveloping curve d does not have a singular point at its point of contact with E_{02}. In that case a stronger condition holds, namely,

(VII′) *The curve E_{03} must not have a point of contact with an enveloping curve d either between* 1 *and* 2 *or at* 2.

If the curve d has a singular point at its contact point with E_{02}, the condition VII still holds and can be proved by other methods; but the case where d touches E at the point 2 requires, as usual, a special investigation.

The equation which defines d on the surface S is

$$\Delta(\sigma, \tau) = \begin{vmatrix} A_\sigma & A_\tau \\ B_\sigma & B_\tau \end{vmatrix} = 0,$$

where A and B are the functions (12) defining the extremals on the surface. If this is satisfied at a point $(\sigma, \tau) = (\sigma_6, 0)$ of the arc E_{12}, the derivative Δ_σ will be different from zero at that point, as is known from the theory of the plane problem applied to the integral (3). The last equation has therefore a solution

$$\sigma = \Sigma(u), \qquad \tau = \mathrm{T}(u),$$

reducing to $(\sigma_0, 0)$ for $u = 0$ and of class C' near that value. The equations of the envelope in space will be

$$(d) \quad x = \xi[\Sigma(u), \mathrm{T}(u)], \quad y = \eta[\Sigma(u), \mathrm{T}(u)], \quad z = \zeta[\Sigma(u), \mathrm{T}(u)].$$

The parameter u is chosen as one of the values $\pm\tau$, but so as to make the positive direction along d the same as that of the surface extremal tangent to it.

For an arbitrary value of u the corresponding value of τ defines an extremal E_{04} of the set (11) which is tangent to the corresponding extremal E_{45} of the set (13) at a point 4 of the curve C as shown in Fig. 2. Further, E_{45} is tangent to d at a point 5, and d_{56} touches E_{03} at the point 6. The families (11), (13), and d are then of the form described in §3 when $u = v$, and the function

$$G(u) = J(E_{04}) + J(E_{45}) + J(d_{56}) + J(E_{63})$$

is a constant, where E_{63} denotes the arc of E_{03} between 6 and 3. An argument precisely like that used for problems in the plane, shows that d_{56} can not be an extremal on the surface, and since

$$G(0) = J(E_{03})$$

it follows that E_{03} can not minimize J if VII is not satisfied.

If the condition VII′ is satisfied together with those postulated at the end of §1, a final necessary condition is the following:

(VIII) *The curve E_{03} must not have a point of contact between* 2 *and* 3 *with an enveloping surface D of the family* (14) *of extremals tangent to S.*

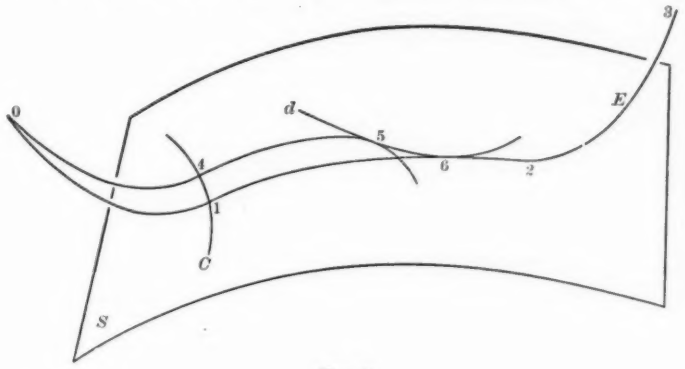

FIG. 2.

Here again the proof will be made for the case where D does not have a singular point at its contact with E_{03}, and the stronger condition holding under those circumstances is

(VIII′) *The curve E_{03} must not have a point of contact with D either between* 2 *and* 3 *or at* 3.

The points on the enveloping surface are defined by the equation

$$\Delta(s,\sigma,\tau)=\begin{vmatrix}\Phi_s & \Phi_\sigma & \Phi_\tau\\ \Psi_s & \Psi_\sigma & \Psi_\tau\\ \mathrm{X}_s & \mathrm{X}_\sigma & \mathrm{X}_\tau\end{vmatrix}=0. \tag{22}$$

On account of Lemma 2 of §2, the determinant is different from zero on the arc E_{23} near the point 2, though it vanishes at 2 itself. If the equation is satisfied at a point 7 of the arc E_{23} for which $(s,\sigma,\tau)=(s_7,s_2,0)$, and further defines a non-singular enveloping surface D, then the argument of Mason and Bliss* shows that there is a curve on D,

$$x=x(u),\qquad y=y(u),\qquad z=z(u), \tag{d}$$

passing through the point 7 for $u=0$, and which is tangent to a one parameter family of the extremals (15) at points defined by equations of the form

$$s=S(u),\qquad \sigma=\Sigma(u),\qquad \tau=\mathrm{T}(u),$$

* See M. and B., §3.

where S, Σ, T reduce to s_7, s_2, 0 for $u = 0$. The parameter u can be so chosen that the positive direction on the curve d coincides with the positive direction of the extremal (15) at the point defined by these functions.

An arbitrarily chosen value u defines an extremal E_{04} of the set (11) tangent to an extremal E_{45} of the set (12) at a point 4 of the curve C; and E_{45} is in turn tangent at a point 5 to the extremal E_{56} defined by the equations

$$(23) \qquad x = \Phi(s, \Sigma, \mathrm{T}), \qquad y = \Psi(s, \Sigma, \mathrm{T}), \qquad z = \mathrm{X}(s, \Sigma, \mathrm{T}).$$

Finally this last curve is tangent to d_{67} at a point 6. When u is set equal to v, the families (11), (13), (23), and d_{67} are of the kind described in §3, and the function

$$G(u) = J(E_{04}) + J(E_{45}) + J(E_{56}) + J(d_{67}) + J(E_{73})$$

is a constant. Since

$$G(0) = J(E_{03})$$

and d can not be an extremal, it follows that E_{03} can not minimize J if VIII is not satisfied.

5. The Existence of a Field

If the arc E_{03} satisfies the stronger set of eight conditions described in §§1, 4, it will surely minimize the integral J with respect to variations sufficiently near to it and lying on one side of the surface S. But before proceeding to the proof of this fact it is necessary to show how a field may be constructed about the minimizing arc.

For this purpose consider first the family (14) which contains the arcs E_{12} and E_{23}, respectively, for the parameter values

$$(\Pi) \qquad \begin{aligned} &s = \sigma, \qquad s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0; \\ &s_2 \leqq s \leqq s_3, \qquad \sigma = s_2, \qquad \tau = 0. \end{aligned}$$

The symbol Π_ϵ will be used to denote the set of points (s, σ, τ) which satisfy either of the following sets of conditions:

$$(\Pi_\epsilon) \qquad \begin{aligned} &\sigma \leqq s \leqq \sigma + \epsilon, \qquad \sigma(\tau) \leqq \sigma \leqq s_2 + \epsilon, \qquad |\tau| \leqq \epsilon; \\ &\sigma + \epsilon \leqq s \leqq s_3 + \epsilon, \qquad |\sigma - s_2| \leqq \epsilon, \qquad |\tau| \leqq \epsilon. \end{aligned}$$

If the constant ϵ is chosen sufficiently small, any two distinct points of Π_ϵ will be transformed by the equations (14) *into distinct points (x, y, z); and in the interior of Π_ϵ the determinant* (22) *will be everywhere different from zero.*

To prove this it will first be shown that in a suitably chosen neighborhood of a point $\pi_0(s_0, \sigma_0, \tau_0)$ satisfying the second of the conditions (II), no two distinct points $\pi(s, \sigma, \tau)$, $\pi'(s', \sigma', \tau')$ of the region Π_ϵ can define

the same point (x, y, z) by means of the equations (14). By Taylor's formula,*

$$\Phi(s', \sigma', \tau') - \Phi(s, \sigma, \tau) = (s' - s) A_1 + (\sigma' - \sigma) A_2 + (\tau' - \tau) A_3 \tag{24}$$

where, for example,

$$A_2 = \int_0^1 \Phi_\sigma (s'', \sigma'', \tau'')\, du, \tag{25}$$

$s'' = s + u(s' - s)$, $\sigma'' = \sigma + u(\sigma' - \sigma)$, $\tau'' = \tau + u(\tau'' - \tau)$, and the other coefficients have similar values. The expression (24) and the similar ones for Ψ and X can vanish simultaneously only when the determinant of the coefficients vanishes. But this is impossible near π_0 since the determinant (22) is different from zero at π_0 and the derivatives of Φ, Ψ, X are continuous.

Near a point π_0 satisfying the first of the conditions II the expression (25) for A_2 can be put in the form

$$A_2 = \int_0^{\sigma'} (s'' - \sigma'') \int_0^{\sigma'} \Phi_{s\sigma} [\sigma'' + v(s'' - \sigma''), \sigma'', \tau'']\, dv\, du,$$

since Φ_σ vanishes when its first two arguments are the same. The point (s'', σ'', τ'') is always in Π_ϵ when π and π' are in Π_ϵ and $0 \leqq u \leqq 1$, and hence $s'' - \sigma''$ is never negative. By a double application of the mean value theorem for a definite integral

$$A_2 = (s_1 - \sigma_1) \Phi_{s\sigma},$$

where the arguments of $\Phi_{s\sigma}$ contain properly chosen intermediate values for u and v, and (s_1, σ_1, τ_1) is the middle point of the segment $\pi\pi'$.

Near a point π_0 satisfying the second of the conditions (II) the difference (24) can therefore be put into the form

$$(s' - s) A_1 + (\sigma' - \sigma)(s_1 - \sigma_1) A_2' + (\tau' - \tau) A_3, \tag{26}$$

where A_1, A_2', A_3 approach continuously the values of Φ_s, $\Phi_{s\sigma}$, Φ_τ at the point π_0 when π and π' approach π_0. This expression and the similar ones for Ψ and X can vanish simultaneously, for π and π' sufficiently near to π_0, only when

$$s' - s = \tau' - \tau = (\sigma' - \sigma)(s_1 - \sigma_1) = 0,$$

since at the point π_0 the determinant (22) is different from zero. But if $s' = s$ and $\tau' = \tau$, the values of σ' and σ must be distinct since π and π' are distinct. The difference $s_1 - \sigma_1$ can not vanish since $s_1 = s = s'$ is greater than the larger of σ and σ' on account of the conditions (Π_ϵ). Hence near π_0 the

* See Jordan, *Cours d'Analyse*, 2d ed., vol. 1, p. 247.

difference (24) and the similar ones for Ψ and X can not all vanish if π and π' are distinct.

Suppose now that it should not be possible to choose a constant ϵ such as is described in the first part of the italicized statement, and consider a sequence $\{\epsilon_k\}$ of decreasing positive constants having zero as its limit. For any choice of k there would exist at least one pair of distinct points,

$$(27) \qquad (s, \sigma, \tau)_k, \qquad (s', \sigma', \tau')_k$$

in the region Π_{ϵ_k} and defining the same point $(x, y, z)_k$ by means of the equations (14). The two sequences (27) would have points of condensation

$$(28) \qquad (s_0, \sigma_0, \tau_0), \qquad (s_0', \sigma_0', \tau_0')$$

necessarily lying in the region Π, and these two points must be the same. Otherwise they would define distinct points of the arc E_{13}, and on account of the continuity of the functions (14) the pairs of values (27) which lie sufficiently near to the values (28) could not define the same point $(x, y, z)_k$. Neither could the pair of sets of values (27) define the same point $(x, y, z)_k$ in case the two condensation points (28) were identical, as has been seen in the paragraphs just preceding.

A similar argument shows that ϵ can be chosen so as to satisfy the last part of the italicized statement. Otherwise there would be a sequence $(s, \sigma, \tau)_k$ of interior points of the respective regions Π_{ϵ_k} at each of which the determinant (22) vanishes. Such a sequence must have a condensation point π_0 in Π. But π_0 could not satisfy the second of the conditions (II), since at such points $\Delta(s, \sigma, \tau)$ is different from zero. Neither could it satisfy the first condition; for near such a point π_0 the value of $\Delta_s(s, \sigma, \tau)$ is different from zero, on account of Lemma 2 and because for $s > \sigma$

$$\Delta(s, \sigma, \tau) = (s - \sigma)\Delta_s[s + \theta(s - \sigma), \sigma, \tau] \neq 0.$$

The points of the sequence $(s, \sigma, \tau)_k$ which condense on π_0 have $s > \sigma$, since they are interior points, and hence the existence of the sequence is contradicted.

If the region Π_ϵ is chosen as described above, its image in the xyz-space, after transformation by equations (14), *is a region $\mathfrak{F}_2$, a continuum and its boundary, whose boundary points are the images of the boundary points of Π_ϵ. The correspondence between $\mathfrak{F}_2$ and Π_ϵ is one-to-one. The single valued functions*

$$(29) \qquad s = s(x, y, z), \qquad \sigma = \sigma(x, y, z), \qquad \tau = \tau(x, y, z)$$

so defined are continuous over $\mathfrak{F}_2$, and in the interior of $\mathfrak{F}_2$ they have continuous derivatives.

The first part of this statement follows at once from some theorems proved

by Bliss,* since the images of the interior and boundary of Π_ϵ are distinct. The preceding paragraphs of this paper show that the correspondence between $\mathfrak{F}_2$ and Π_ϵ is one-to-one. An interior point of $\mathfrak{F}_2$ corresponds to an interior point of Π_ϵ at which $\Delta(s, \sigma, \tau) \neq 0$, and the fundamental theorem of the implicit function theory shows at once that the functions (29) are of class C' at such points. If a sequence of points $(x, y, z)_k$ of $\mathfrak{F}_2$ approaches a boundary point (x', y', z') of $\mathfrak{F}_2$, the corresponding values $(s, \sigma, \tau)_k$ defined by the equations (29) must approach the values (s', σ', τ') which define (x', y', z') by means of equations (14), as may readily be seen from the continuity of the functions $\Phi, \Psi, \mathbf{X}$. Hence the functions (29) are continuous everywhere in $\mathfrak{F}_2$.

There exists a field $\mathfrak{F}_1$ about the arc E_{01} through each point of which there passes a unique extremal of the set (6) *defined by equations of the form*

$$\begin{aligned} s &= s(x, y, z), \qquad u = u(x, y, z), \\ v &= v(x, y, z), \qquad w = w(x, y, z). \end{aligned} \tag{30}$$

These functions are continuous everywhere in $\mathfrak{F}_1$ and have continuous derivatives except at the point 0.†

The symbol $\mathfrak{F}_{12}$ will be used to denote the set of points (x, y, z) consisting of the interior of $\mathfrak{F}_2$, the points of S in $\mathfrak{F}_2$, and the points interior to $\mathfrak{F}_1$. Let V be a continuous curve

$$(V) \qquad x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t) \qquad (t_0 \leqq t \leqq t_3)$$

consisting of a finite number of non-singular arcs of class C', joining the points 0 and 3, and lying in the region $\mathfrak{F}_{12}$ on the same side of S with the arc E_{01}.

If $\mathfrak{F}_1$ is taken sufficiently small the only points where V can enter $\mathfrak{F}_2$ are on the boundary B of $\mathfrak{F}_2$ which corresponds to the parameter values

$$\sigma \leqq s < \sigma + \epsilon, \qquad \sigma = s(t), \qquad |\tau| < \epsilon. \tag{31}$$

A neighborhood of the point 1 can be chosen so small that the only boundary points of $\mathfrak{F}_2$ in it are points of the surface S and points on the boundary B defined by the conditions (31). Further, $\mathfrak{F}_1$ can be taken so small that the only points common to it and $\mathfrak{F}_2$ are in the same neighborhood. Under these circumstances the curve V can enter $\mathfrak{F}_2$ only at points of B or at points of S

* *The Princeton Colloquium Lectures on Mathematics*, p. 38.

† For the existence of the field in the neighborhood of the point 0, see Bliss and Mason, *Fields of extremals in space*, these Transactions, vol. 11 (1910), p. 328; also references at the beginning of the same paper; also Szücs, Mathematische Annalen, vol. 71 (1911), p. 380. The existence of the field about the rest of the arc follows by the usual methods. A proof analogous to that just preceding in the text is more elegant than any o these.

distinct from B. The latter are, however, excluded since V lies entirely on one side of S near the point 1.

6. Sufficient Conditions for a Minimum

It is proposed to prove in this section that an arc E_{03} of the kind described at the end of §5, will give the integral J a value at least equal to $J(E_{03})$. To each value of t defining a point 6 of V_{03} in $\mathcal{F}_2$, there will correspond a unique curve

$$E_{06} = E_{04} + E_{45} + E_{56},$$

shown in Fig. 1, where E_{04}, E_{45}, E_{56} belong respectively to the families (11), (13), and (14); while if t defines a point 6 which does not lie in $\mathcal{F}_2$ it will determine a unique extremal E_{06} of the family (6).

The integral $J(E_{06})$, is a continuous function of t which has the derivative

$$\frac{d}{dt}J(E_{06}) = F_{x'}\bar{x}' + F_{y'}\bar{y}' + F_{z'}\bar{z}'\,|^{6}. \tag{32}$$

In this expression the arguments of the derivatives of F are the values (x, y, z, x', y', z') which define the point and direction of E_{06} at the point 6, while $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ are the derivatives at the same point of the functions (V) defining V.

To prove this consider the one parameter family of curves E_{06} determined by the points of V_{03} interior to $\mathcal{F}_2$. It is determined by substituting the functions defining V in the expressions (29) for s, σ, τ in terms of x, y, z. The functions $s(t)$, $\sigma(t)$, $\tau(t)$ so determined are of class C' near a value of t defining an interior point of $\mathcal{F}_2$ unless t is one of the exceptional values at which V has a corner point. By substituting $\sigma(t)$, $\tau(t)$ in the equations (11), (13), and (14), the three families E_{04}, E_{45}, E_{56} are determined with properties similar to those of the auxiliary theorems of §3. It follows readily that the derivative of $J(E_{06})$ has the value given in the theorem.

The argument is the same for a point 6 in $\mathcal{F}_1$ but not in $\mathcal{F}_2$, except that the equations whose solutions determine the family E_{06} are then

$$x(t) = \phi(s, u, v, w), \qquad y(t) = \psi(s, u, v, w), \qquad z(t) = \chi(s, u, v, w),$$
$$u^2 + v^2 + w^2 = 1.$$

When the point 6 lies on the surface S at the point 0, or on the boundary B, it is more difficult to show the existence of the derivative. But the desired result can be obtained by the use of expansions of the functions (14) similar to (26), by means of the properties of the functions (6) at the point 0 described in the paper referred to in the preceding section, and by special considerations at a point of B.

* At corner points of the curve V this should be thought of as a forward derivative.

If the stronger set of eight properties described in the preceding sections is satisfied by the arc E_{03}, then

$$J(E_{03}) < J(V_{03}) \tag{33}$$

for any curve V_{03} distinct from E_{03} in the field $\mathfrak{F}_{12}$ and on the same side of S with the arc E_{03}. It is understood that V_{03} consists of a finite number of non-singular arcs of class C'.

For consider the continuous function

$$W(t) = J(E_{06}) + J(V_{63}),$$

where 6 is the point of V determined by the parameter value t. With the help of the last theorem but one, the derivative of $W(t)$ exists except at a finite number of values of t, and has itself the value

$$W'(t) = -E(x, y, z; x', y', z'; \bar{x}', \bar{y}', \bar{z}'). \tag{34}$$

On account of IV and the properties of the function E this is always negative except when the positive directions along V_{63} and E_{06} coincide at the point 6. Furthermore

$$W(t_0) = J(V_{03}), \qquad W(t_1) = J(E_{03}).$$

The inequality (33) must therefore be satisfied unless the positive directions along E_{06} and V_{63} coincide at every point 6 of V where the derivative (34) is well defined.

If the positive directions along V coincide with those of E_{06} at every point of V, then V_{03} and E_{03} must be coincident. The proof is somewhat complicated but may be made with the help of the following lemma:

LEMMA 3. *Any arc of V defined by an interval $t' \leqq t \leqq t''$ and containing no points of S or B, is an extremal of one of the families* (6), (14) *along which s increases from s' to s'' as t traverses the interval t' t''. An arc of V lying entirely on the surface S must be an arc of one of the surface extremals* (13) *with similar properties.*

An arc of V with which the lemma is concerned must lie in the interior of $\mathfrak{F}_1$ or $\mathfrak{F}_2$, or else on the surface S. The proofs in the three cases are quite similar, so that it will be sufficient to suppose the arc V to lie entirely on S. The values $s = \sigma$, σ, τ are then continuous functions $\sigma(t)$, $\tau(t)$ defined over the interval $t' \leqq t \leqq t''$ by equations (V) and (29), and they satisfy the equations

$$\begin{gathered} x(t) = \xi[\sigma(t), \tau(t)], \qquad y(t) = \eta[\sigma(t), \tau(t)], \\ z(t) = \zeta[\sigma(t), \tau(t)]. \end{gathered} \tag{35}$$

At least one of the determinants of the matrix

$$\left\| \begin{matrix} \xi_\sigma & \eta_\sigma & \zeta_\sigma \\ \xi_\tau & \eta_\tau & \zeta_\tau \end{matrix} \right\|$$

is different from zero at every value of t on account of VII′ and the relations (13). Hence the theory of implicit functions applied to a suitably chosen pair of the equations (35) shows that $\sigma(t)$ and $\tau(t)$ are of class C'. If the direction of V coincides with that of the surface extremal at each point of the arc in question, then

$$x'(t) = \kappa\,\xi_\sigma, \qquad y'(t) = \kappa\,\eta_\sigma, \qquad z'(t) = \kappa\,\zeta_\sigma \qquad (\kappa > 0),$$

and by differentiating equations (35) the relations

$$\sigma'(t) = \kappa, \qquad \tau'(t) = 0$$

are readily deduced. Hence the lemma is true for an arc on S, and for the other two cases by similar arguments.

LEMMA 4. *If t_4 is a lower bound of values of t such that for $t_4 \leqq t \leqq t_3$ the curve V_{03} coincides with E_{03}, then no value $t' < t_4$ can define a point of V interior to $\mathfrak{F}_2$, or a point of V on the surface S and distinct from B.*

There must be a lower bound t_4 of the kind described since near the point 3 the curve V_{03} coincides with an extremal of the field passing through the point 3, according to Lemma 3, and E_{03} is the only such extremal.

If t' defined a point interior to $\mathfrak{F}_2$, then there would be a first value $t'' > t'$ such that the arc $t'\,t''$ of V would be entirely interior to $\mathfrak{F}_2$ and distinct from E_{03} except at the value t''. The value t'' would necessarily define a point either of B, or S, or E_{23}. But the arc $t'\,t''$ of V would in that case be an extremal arc along which s increases with t from a value s' to a value s''. This is, however, impossible, since any such arc which is distinct from B, S, E_{23} for $s = s'$ retains this property for all values $s > s'$.

With the help of this result and by similar arguments it follows that t' could not define a point of S distinct from B.

The identity of V_{03} and E_{03} now follows without difficulty. The value $t = t_4$ must in fact define a point 4 on the arc E_{01}, since in case this is not true there would certainly be values $t' < t_4$ satisfying the conditions of Lemma 4. The arc $t_0\,t_1$ of V then lies entirely in the field $\mathfrak{F}_1$, and no point of it can be distinct from E_{01}. Otherwise there would be an extremal arc of the field $\mathfrak{F}_1$ distinct from E_{01} whose end point, as s increased toward the parameter value defining it, would lie on E_{01}. There is, however, no such arc.

ON A METHOD OF COMPARISON FOR TRIPLE-SYSTEMS*

BY

LOUISE D. CUMMINGS

INTRODUCTION

1. If n elements $1, 2, 3, \cdots n$ can be distributed in triads in such a way that every pair of elements appears in one and only one triad, the totality of triads forms a triple-system of n elements. Reiss† has shown that it is possible to form a triple-system of n elements provided n is of the form $6m + 1$ or $6m + 3$.

Different methods for constructing triple-systems have been given by Reiss,† Netto,‡ Heffter,§ and E. H. Moore;|| but methods for testing the non-congruency of these systems when formed are lacking. The group of substitutions that transform a triple-system into itself has hitherto been adopted as the abstract mark of the system, and Zulauf¶ has shown, by a consideration of the groups, that the four systems on 13 elements of Kirkman, Reiss, De Vries, and Netto are reducible to two incongruent systems whose groups are different. In the present paper it is shown however that for $n = 15$ two incongruent triple-systems, Δ_{15}, may have the same group.

By a simple process non-congruent triple-systems are constructed. These are used to illustrate a new method of comparison, by means of a new sort of abstract marks. This method requires no knowledge of the group but incidentally facilitates its determination. No exhaustive determination of every Δ_{15} which may be obtained by this process is here undertaken, but the 24 systems discussed include 12 apparently not hitherto constructed.

APPLICATION OF THE METHOD TO A REISS TRIPLE-SYSTEM ON 15 ELEMENTS

2. A triple-system formed by the Reiss method** is shown in the accompanying table, which gives 28 triples of the system; the remaining 7, formed

* Presented to the Society, April, 1914.

† *Ueber eine Steinersche kombinatorische Aufgabe*, Crelle's Journal, vol. 56 (1859), pp. 326–344.

‡ *Zur Theorie der Tripelsysteme*, Mathematische Annalen, vol. 42 (1892), pp 143–152.

§ *Ueber Tripelsysteme*, Mathematische Annalen, vol. 49 (1897), pp. 101–112.

|| *Concerning Triple Systems*, Mathematische Annalen, vol. 43 (1893), pp. 271–285.

¶ *Ueber Tripelsysteme von 13 Elementen*. Dissertation. Marburg, 1897.

** M. Reiss, l. c.

from the elements which head the columns, are abd, acg, aef, bce, bfg, cdf, deg.

TABLE I

a	b	c	d	e	f	g
87	86	85	84	83	82	81
62	71	76	75	74	73	72
53	52	31	61	65	64	63
41	43	42	32	21	51	54

This system may be arranged in the following rectangular array. Each element heads one column; below it are placed the 7 pairs of elements that occur with it in triads of the system.

a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
87	86	85	84	83	82	81	$a4$	$a6$	$a5$	$a1$	$a3$	$a2$	$a8$	$a7$
62	71	76	75	74	73	72	$b7$	$b5$	$b4$	$b3$	$b2$	$b8$	$b1$	$b6$
53	52	31	61	65	64	63	$c3$	$c4$	$c1$	$c2$	$c8$	$c7$	$c6$	$c5$
41	43	42	32	21	51	54	$d6$	$d3$	$d2$	$d8$	$d7$	$d1$	$d5$	$d4$
bd	ad	ag	ab	af	ae	ac	$e2$	$e1$	$e8$	$e7$	$e6$	$e5$	$e4$	$e3$
cg	ce	be	cf	bc	bg	bf	$f5$	$f8$	$f7$	$f6$	$f1$	$f4$	$f3$	$f2$
ef	fg	df	eg	dg	cd	de	$g8$	$g7$	$g6$	$g5$	$g4$	$g3$	$g2$	$g1$

To examine the relations existing among the elements of the system, consider any triple, as $a87$ and the 21 pairs of elements which are united with the 3 elements of this triple.

a	8	7
87	$a7$	$a8$
62	$b6$	$b1$
53	$c5$	$c6$
41	$d4$	$d5$
bd	$e3$	$e4$
cg	$f2$	$f3$
ef	$g1$	$g2$

Begin with a pair in the column headed a, for example 62, written thus: call 6 the first element and 2 the second element of this pair; proceed from one column to the next in cyclical order, selecting in successive columns the pair which contains the second element of the preceding pair; continue this process until the pair to be selected from the first column is the initial pair. For the triple $a87$ the pairs would then appear as follows: 62, $2f$, $f3$, 35, $5c$, $c6$, 62, $2f$, the pairs repeating after each of the three columns has been examined twice. The occurring pairs, the periodicity, and the fact that the *period* consists of two rounds of the three columns can be shown conveniently by the notation $62f/35c/6$. The bar between two elements indicates the completion of one round of the three columns. Such a succession of pairs, repeating after 2 rounds of the columns, forms a *sequence* of period 2.

Since the elements in the pair 62 can be written either as 62 or 26, examine next this same pair with the two elements written in reverse order. Beginning with the pair 26 the pairs are selected from the columns as follows: 26, $6b$, $b1$; 14, $4d$, $d5$; 53, $3e$, $e4$; 41, $1g$, $g2$; 26, showing that after 4 rounds of the columns, the first column is again reached with the initial pair, or in sequence notation $26b/14d/53e/41g/2$.

Apply this process to the remaining pairs 53, 41, *bd*, *cg*, *ef*, of column *a*. The pair 53 appears in the sequence of period 4 and does not require further examination, since 53 as initial pair would furnish the sequence 53*e*/41*g*/26*b* /14*d*/5, which is identical with the above sequence of period 4 except as to starting point; hence any pair which has appeared in a sequence can be omitted from further examination. The pair 41 appears in the sequence of period 4; the pair *bd* furnishes the sequence *bd*4/*ef*2/*gc*5/*db*6/*cg*1/*b*, of period 5; the pair *fe* leads to the sequence *fe*3/*f* of period 1; the pair 87 to the sequence 87*a*/8. The pair 78 leads from the body of the first column up to the element 8 at the head of the second column and the process cannot be continued. Therefore in examining the pairs of a column we omit the pair which, with the element at the head of the column, completes the triple under examination.

For the triple *a*87 the sequences are 62*f*/35*c*/6, 26*b*/14*d*/53*e*/41*g*/2, *bd*4/*ef*2/*gc*5/*db*6/*cg*1/*b*, *fe*3/*f*; and all possible pairs for the column *a*, namely 62, 26, 53, 35, 41, 14, *bd*, *db*, *cg*, *gc*, *ef*, *fe*, appear in these sequences. The number of rounds of the columns required to close a sequence (the period) is recorded for each sequence belonging to the triple under examination, and we shall call these numbers the *index* of that triple, and the longest sequence the *major sequence.*

The index of the triple *a*87 consists of the numbers 2, 4, 5, 1. The index shows that the sequences for the triple *a*87 consist of a sequence closed in two cycles of three elements, hence containing two triads, a second sequence containing four triads, a third sequence containing five triads and a fourth sequence containing one triad. In any column twelve pairs must be considered and each pair occurs in one and only one round of the columns and therefore in one and only one triad in the sequences. Hence for a triple-system on 15 elements, the sum of the periods for every index is 12. Indices of the types 111117, 111144, 66 are written in the abbreviated form $1^5 7$, $1^4 4^2$, 6^2.

The sequences are closed cycles, in which any element may be selected as initial element and the sequence may be read in direct or reverse order. The sequences for *a*87, 87*a*, and 8*a*7 illustrate the following

THEOREM. *The index of every triple in a given triple-system is invariant under all permutations of its triad.*

ANALYSIS OF THE MAJOR SEQUENCES

3. An element which appears once, twice, three times in a sequence is designated as a single, double, triple point, respectively, for that sequence. It is evident that a sequence of period 1 contains no double point and hence that no duplicate element occurs in a triad of a sequence.

LEMMA. *A sequence of period greater than one contains no repetition in any quintad of elements.*

x	y	z
ab	bc	cd
de		

Let the sequence for xyz begin in the column x with the initial pair ab, and for convenience number the places in the sequence. The elements in places 1, 2, and 3 may be a, b, and c, but the element in the place 4 is not a, since that would close the sequence. No repetition occurs in a triad of a sequence, hence the element in place 4 is neither b nor c, and therefore must be a new element d. The element to fill place 5 forms with d a pair in column x and so is neither a nor b; since this required element forms a triad with c, d, it must be a new element e. Hence no repetition occurs in a quintad of elements. By applying the above Lemma and reading the sequence forward and backward we obtain the following

THEOREM. *A sequence of period equal to or less than three contains no multiple point.*

4. This process applied to the 35 triples of the Reiss Δ_{15} gives indices and sequences only of certain types, namely:

	TRIPLE	INDEX	SEQUENCES
(1)	abd	$1^5\,7$	871/625/786/143/268/417/534/8, 352/3, cgf/c, gce/g, efg/e, fec/f;
(2)	$a87$	1245	$62f/35c/6$, $26b/14d/53e/41g/2$, $bd4/ef2/gc5/db6/cg1/b$, $fe3/f$;
(3)	$a41$	6^2	$87e/26f/53b/78d/62c/35g/8$, $bd8/gc2/ef6/db3/cg5/fe7/b$.

The indices have separated the 35 triples of the Reiss system into 3 classes, (1), (2), and (3). This separation into classes will be found to simplify the problem of determining the group of the system. For every substitution which converts the system into itself must permute among themselves the triples of a class, since formation of cycles depends on identity or difference of elements in the different columns and these are not changed by any substitution.

The 35 triples arranged in classes according to their indices are shown in the following table.

Classification of the Reiss Δ_{15} by indices

$1^5\,7$			1245							6^2		
abd	bce	deg	$a26$	$b25$	$c24$	$d23$	$e38$	$f28$	$g27$	$a14$	$d16$	$g18$
acg	bfg		$a35$	$b34$	$c58$	$d48$	$e47$	$f37$	$g36$	$b17$	$e12$	
aef	cdf		$a78$	$b68$	$c67$	$d57$	$e56$	$f46$	$g45$	$c13$	$f15$	
(1)			(2)							(3)		

In the major sequence 871/625/786/143/268/417/534/8 of the triple *abd*, the second 8 enters in the seventh place after the first 8, the third 8 in the seventh place after the second 8, and the first 8 in the seventh place after the third 8; we denote this by the symbol $8(7,7,7)$; with this notation the elements of the sequence are characterized as follows: $8(7,7,7)$, $7(5,11,5)$, $1(7,7,7)$, $6(5,5,11)$, $2(8,13)$, $5(13,8)$, $4(5,5,11)$, $3(8,13)$. For a sequence of period 7 containing 21 elements the symbols $(5,11,5)$, $(5,5,11)$, $(11,5,5)$ are identical and $(8,13) \equiv (13,8)$. Hence this sequence contains two distinct types of triple points and one type of double points.

The analysis of the major sequences of the triples of class (1) is given in the following table:

Class (1)	(7,7,7)	(5,11,5)	(7,7,7)	(5,5,11)	(8,13)	(13,8)	(5,5,11)	(8,13)
abd	8	7	1	6	2	5	4	3
bce	4	3	1	2	5	8	7	6
cdf	7	6	1	5	8	4	3	2
deg	3	2	1	8	4	7	6	5
efa	6	5	1	4	7	3	2	8
fgb	2	8	1	7	3	6	5	4
gac	5	4	1	3	6	2	8	7

Determination of the group for the Reiss Δ_{15}

5. Any operation of the group that leaves a triple-system invariant can only transform any triad and its sequences into itself, or into another of the same class. Therefore to find the group, we examine first for substitutions that transform into itself one of the longest sequences, and secondly for those that transform it into the remaining sequences of its class. Substitutions when determined must be tested on the whole triple-system.

An examination of the table of indices for this system shows that the 15 elements do not enter symmetrically as members of the triads in the classes, for example, the 7 triads in class (1) contain only the elements a, b, c, d, e, f, g. Since only those elements may be permuted which occur the same number of times in a class, the enumeration of the appearances of each element in the classes, as in the following table, shows the possible sets of transitive elements.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
(1)	3	3	3	3	3	3	3								
(2)	3	3	3	3	3	3	3		6	6	6	6	6	6	6
(3)	1	1	1	1	1	1	1	7	1	1	1	1	1	1	1

The systems of transitivity for the group are therefore a, b, c, d, e, f, g; 1; 2, 3, 4, 5, 6, 7, 8.

The substitutions may be determined from any class in the system, but most easily from the class containing the longest sequences since these exhibit

more repetitions of the elements. In the present case class (1) is therefore selected.

(1) Examine for substitutions to transform the triple *abd* into itself. The major sequence of *abd* is 871625786143268417534, and a consideration of the possible rotations of this sequence into itself will determine the substitutions which transform the triple *abd* into itself. The element 1 forms a transitive system; hence a rotation of period 3 given by (1) (647) (235) (8) transforms this sequence into itself. The triple-system will now enable us to complete the substitution, for if two elements in one triple transform into two elements in a second triple, the third element in the first triple transforms into the third element in the second triple, for example, 18*g* transforms into 18*g*. An application of both cycles (235) and (647) to the two triples 26*a* and 24*c* gives respectively 26*a*, 34*b*, 57*d*; 24*c*, 37*f*, 56*e* and shows for the required substitution additional cycles (*abd*) and (*cfe*). Hence the substitution (*abd*) (*cfe*) (*g*) (1) (235) (476) (8) transforms *abd* and its sequences into itself.

Any rotation of period 2 which transforms the sequence of *abd* into itself must keep fixed one of the elements 1 and interchange elements at equal distances from this 1, for example, (1) (46) (38) (27) $\cdots$, but in the sequence 3 is a double point and 8 is a triple point and it is not possible to convert the sequence into itself by the interchange of two points of different multiplicities; hence no substitution of period 2 exists which will convert this sequence into itself. Therefore only a sub-group of order 3 transforms the triple *abd* into itself.

(2) Examine for substitutions to transform the triple *abd* into the other triples in its class. The table of class (1) exhibits the cycles (8473625) (1). If now we apply these two cycles to any triple in the system, as *a*14, then the triples of the system appear in the following succession: 14*a*, 17*b*, 13*c*, 16*d*, 12*e*, 15*f*, 18*g*, showing the cycle (*abcdefg*) and this completes the substitution $s_2 \equiv (abcdefg)\ (1)\ (2584736)$ which transforms *abd* in succession into every other triple in its class. Therefore the group for the Reiss system is of order 21 and is generated by

$s_1 \equiv (abd)\ (cfe)\ (g)\ (1)\ (235)\ (476)\ (8)$, $s_2 \equiv (abcdefg)\ (1)\ (2584736)$.

The method of construction employed by Reiss involves a cyclic substitution of order 7, hence the appearance of s_2 was to be expected; but it did not show the substitution of order 3.

Construction of new systems

6. The seven triples containing only the elements which head the columns in the seven-by-four array, Table I, §2, form a triple-system on 7 elements, which is designated as a *head* for this array. The seven triples in the head

and the twenty-eight triples in the array make up the thirty-five triples in the system. New systems may be formed by retaining the twenty-eight triples of the seven-by-four array and changing the seven triples of the head or by changing the seven-by-four array. An index determined from a seven-by-four array is designated as a *partial* index to distinguish it from a complete index determined from a fifteen-by-seven array.

Formation of new systems by changing the head of the seven-by-four array

7. From the seven elements a, b, c, d, e, f, g thirty-five triples can be formed. We determine for each of these triples its partial index from the seven-by-four array I. For example:

Triple	Index	Sequences
abc	$1^2, 2, 4$	786/7, 871/352/417/625/8, 268/534/2, 143/1;
abd	1, 7	871/625/786/143/268/417/534/8, 352/3.

Table of partial indices

$1^2, 2, 4$							1, 7				
abc	*ace*	*adf*	*bcd*	*bdg*	*cde*	*cfg*	*abd*	*acg*	*bde*	*aef*	*deg*
abe	*acf*	*adg*	*bcf*	*bef*	*cdg*	*efg*	*abf*	*bce*	*cdf*	*aeg*	*dfg*
abg	*ade*	*afg*	*bdf*	*beg*	*ceg*	*def*	*acd*	*beg*	*cef*	*bfg*	

Select any triple, as *abd*. Construct all the triple-systems on the seven elements a, b, c, d, e, f, g, which contain *abd*, and retain one of each type. Six systems Δ_7 can be formed, but the sets of partial indices of four of those systems are identical. Hence this triple furnishes only three types of head for the seven-by-four array, namely A, B, and C, as in the table.

A		*B*		*C*	
abd	1, 7	*abd*	1, 7	*abd*	1, 7
acg	1, 7	*acg*	1, 7	*ace*	$1^2, 2, 4$
aef	1, 7	*aef*	1, 7	*afg*	$1^2, 2, 4$
bce	1, 7	*bcf*	$1^2, 2, 4$	*bcf*	$1^2, 2, 4$
bfg	1, 7	*beg*	$1^2, 2, 4$	*beg*	$1^2, 2, 4$
cdf	1, 7	*dce*	$1^2, 2, 4$	*cdg*	$1^2, 2, 4$
deg	1, 7	*dfg*	1, 7	*def*	$1^2, 2, 4$

Since triples with different partial indices may give different types of heads, we examine in the same way the triple *abe*; this furnishes 4 heads of type C and 2 heads of type B.

The 35 triples formed of the elements a, b, c, d, e, f, g contain only 2 partial indices, 1^2 24 and 17, and all triple-systems of seven elements have been formed for one triple selected from each class of partial indices, and three types of heads A, B, and C have appeared. The twenty-eight triples in the seven-by-four array completed with any one of these heads form a triple-system on 15 elements.

The array, Table I, completed by these three heads, is denoted by IA, IB, IC; of these IA is the Reiss system already investigated.

	a	b	c	d	e	f	g
I	87	86	85	84	83	82	81
	62	71	76	75	74	73	72
	53	52	31	61	65	64	63
	41	43	42	32	21	51	54
IA	*abd*	*acg*	*aef*	*bce*	*bfg*	*cdf*	*deg*
IB	*abd*	*acg*	*aef*	*bcf*	*beg*	*cde*	*dfg*
IC	*abd*	*ace*	*afg*	*bcf*	*beg*	*cdg*	*def*

To show that the 3 systems IA, IB, IC, are non-congruent, it is necessary to write each system in the form of the fifteen-by-seven array and to obtain the indices for the 35 triples of each system; since it is possible that triples with different partial indices may have the same complete index and *vice versa*.

No two congruent systems have been admitted, but it does not follow that non-congruent systems have not been excluded, and in an exhaustive determination of all systems Δ_{15} every head must be applied to the seven-by-four array, and the indices for each system determined from its fifteen-by-seven array.

8. Classification of IB by indices and transitive elements

$1^6\,24$	$1^5\,7$	$1^3\,234$	$1^2\,2^2\,4$	$1^2\,46$	1245			$2^3\,6$	$2^2\,4^2$		6^2
bcf	*abd*	*d*23	*b*25	*b*17	*a*87	*b*68	*f*28	*d*16	*b*34	*d*48	*a*14
beg	*acg*	*f*37	*c*67	*c*13	*a*26	*e*56	*g*36	*f*15	*c*24	*f*46	
cde	*aef*	*g*27	*e*38	*e*12	*a*35	*c*58	*d*57	*g*18	*e*47	*g*45	
	dfg										
(1)	(2)	(3)	(4)	(5)		(6)		(7)	(8)		(9)

The enumeration of the entrances of the elements in the 9 classes shows that for the group of the system the sets of transitive elements are a; b, c, e; d, f, g; 1; 2, 3, 7; 5, 6, 8; 4. These sets of transitive elements subdivide the classes into sets of triples which are not transformable into one another by operations of the group of the system; the subdivisions are shown by lines separating the triples in a class, and the system IB contains 13 subdivisions. The group is generated by $s \equiv (a)\,(bec)\,(dfg)\,(1)\,(237)\,(586)\,(4)$, and is of order 3.

9. Classification of IC by indices and transitive elements

$1^6\,24$		$1^5\,7$	$1^3\,234$	$1^2\,2^2\,4$		$1^2\,46$		1245		$2^3\,4^2$	$23^2\,4$	6^2
ace	*beg*	*abd*	*c*58	*a*35	*g*27	*a*14	*c*13	*a*78	*c*24	*a*26	*c*67	*g*18
def	*cdg*		*e*38	*b*25	*g*36	*b*17	*e*12	*b*68	*e*56	*b*34	*e*47	
bcf	*afg*		*f*28	*d*23	*g*45	*d*16	*f*15	*d*48	*f*37	*d*57	*f*46	
(1)		(2)	(3)	(4)		(5)		(6)		(7)	(8)	(9)

The sets of transitive elements for the system are $a, b, d; c, e, f; g; 1; 2, 3, 5; 4, 6, 7; 8$. These with the indices separate the system into 13 non-permutable subdivisions. The group is generated by $s \equiv (adb)\,(cef)\,(g)\,(1)\,(253)\,(467)\,(8)$, and is of order 3.

10. A comparison of I*B* and I*C* shows that the groups of these two non-congruent systems are the same. For these groups are generated respectively by $s \equiv (a)\,(bec)\,(dfg)\,(1)\,(237)\,(4)\,(586)$, $t \equiv (adb)\,(cef)\,(g)\,(1)\,(253)\,(467)\,(8)$, and are abstractly the same, since each of these substitutions permutes 12 elements, in 4 cycles, each of order 3, and leaves the remaining 3 elements fixed. But the system I*B* contains 4 triples with index $1^5\,7$, while the system I*C* contains only 1 triple with index $1^5\,7$. Now in a given system the index of a triple is invariant under all the substitutions of the symmetric group; two systems are, therefore, certainly incongruent if their sets of indices are unlike, whether their groups are different or identical.* *Hence no substitution exists which converts* I*B into* I*C, and the two systems are non-congruent. That is, as pointed out in* §1, *the same group belongs to two non-congruent systems.*

Up to the present time the group has been considered as the abstract mark of a system. It is true that two systems with different groups are distinct, but as shown above, the converse is not true, a fact which apparently has not been suspected before.

Formation of new systems by changing the fundamental seven-by-four array

11. If a seven-by-four array shows 3 columns of the types x, y, and z containing the 3 sets of pairs 12, 34: 13, 24: 14, 23 which can be formed from the four elements 1, 2, 3, 4, the four elements constitute a *quadrangular array* which may be denoted by the symbol $\left[\begin{smallmatrix}12\\34\end{smallmatrix}\right]$. The remaining four elements 5, 6, 7, 8 must necessarily form also a quadrangular array; hence quadrangular arrays enter into a seven-by-four array always by couples, and it is easily shown that a seven-by-four array must contain 0, 2, 6, or 14 quadrangular arrays; — a fact hitherto not noticed, but of considerable importance in the following discussion.

x	y	z
12	13	14
34	24	23
56	57	58
78	68	67

An examination of every combination of two pairs occurring in the 7 columns of Table I shows that this seven-by-four array contains no quadrangular array.

Let us form, then, a new fundamental Table II, which shall contain two and only two of these quadrangular arrays. In constructing columns suitable for a seven-by-four array, the first pair in the column may be selected in 7 ways (since with the first element we may unite any one of the remaining

* The truth of the converse is not here asserted, though it seems probable.

7 elements), the second pair in 5 ways, the third pair in 3 ways, and the fourth pair only in 1 way; hence 105 suitable columns may be formed.

From the 105 columns we select 3 columns a, b, c, containing the 2 quadrangular arrays $\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}5&6\\7&8\end{smallmatrix}\right]$. The remaining 4 columns d, e, f, g, must be selected so that no more quadrangular arrays are introduced, and the entering columns must not contain any pair that appears in the columns already in the table.

TABLE II

a	b	c	d	e	f	g
87	86	85	84	83	82	81
65	75	76	73	72	71	74
43	42	41	61	64	63	62
21	31	23	25	15	45	53

The Table II shows one of the several possible ways of constructing a seven-by-four array of the desired type. The determination, from the seven-by-four array II, of the partial indices for the 35 triples which may be formed from the 7 elements a, b, c, d, e, f, g, shows 5 different partial indices. The re-heading process is now applied to Table II. A triple of each index furnishes 6 systems Δ_7. By the selection of 1 system of each type from these 30, the 6 different heads A, B, C, D, E, F shown in the following table are obtained for the seven-by-four array II.

	a	b	c	d	e	f	g					
II	87 65 43 21	86 75 42 31	85 76 41 23	84 73 61 25	83 72 64 15	82 71 63 45	81 74 62 35	NUMBER OF TRIPLES OF THE HEAD WITH PARTIAL INDICES				
								1^8	$1^4 2^2$	$1^2 4$	2^4	4^2
A	bde	abc	bfg	adf	cdg	aeg	cef	1	0	4	2	0
B	bde	abf	bcg	acd	dfg	aeg	cef	0	0	2	2	3
C	bde	abf	bcg	adg	cdf	ace	efg	0	2	1	1	3
D	bde	abg	bcf	acd	dfg	aef	ceg	0	2	1	2	2
E	bde	bac	bfg	dag	cdf	aef	ceg	1	4	2	0	0
F	bde	abg	bcf	adf	cdg	ace	efg	0	0	2	3	2

If a system Δ_{15} contains more than one Δ_7, then for every such contained Δ_7 there is an arrangement of the Δ_{15}, showing this Δ_7 as a head to a suitable seven-by-four array. These different arrangements may furnish dissimilar partial indices. For example, the system IIA contains three systems Δ_7 formed of the elements a, b, c, d, e, f, g; $a, b, c, 1, 2, 3, 4$; and $a, b, c, 5, 6, 7, 8$. The second of these heads, placed over the seven-by-four array formed of the 28 remaining triples of the system IIA, gives for the triples $abc, a43, a12, b42, b31, c41, c23$, respectively, the following partial indices 1^8, $1^4 2^2$, $1^4 2^2$, 2^4, 2^4, 4^2, 4^2. Hence IIA shows dissimilar sets of partial indices for different heads. Therefore if a system Δ_{15} contains more than one Δ_7, a determination of the *partial indices* of the seven triples of one head is not a sufficient characterization of the system.

To determine if the six systems with fundamental Table II are distinct, we compare the indices of the thirty-five triples of each system, since systems with different sets of indices are incongruent. The indices for the 6 systems with fundamental seven-by-four array II are shown in the following table:

II	1^{12}	$1^8 2^2$	$1^6 24$	$1^5 2^2 3$	$1^4 2^4$	$1^4 26$	$1^4 4^2$	$1^3 2^2 5$	$1^3 234$	$1^2 2^3 4$	$1^2 2^2 3^2$	$1^2 46$	1236
A	1		12		10						12		
B			2		2	4	3	4	4	4	8		4
C		2	1	6	5		3	4	4		4	2	4
D		2	1	6	4		2	6	2	2	8	2	
E	1	10	4	8	6		2				4		
F			2		5	4	2		4	4	10		4

This table shows 6 distinct systems differing from the 3 systems with fundamental seven-by-four array I.*

The sets of transitive elements for II*A* are $a, b, c; d, f, 2, 4, 5, 8; e, g, 1, 3, 6, 7$; these with the classes separate the 35 triples of the system into 6 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (ab)(d2f4)(e1g3)(58)$, $s_3 \equiv (abc)(d48)(e17)(f25)(g36)$, and is of order 24.

The sets of transitive elements for II*B* are $a; b, c; d, f; e; g; 1, 3, 6, 7; 2, 4, 5, 8$; these with the classes separate the system into 13 non-permutable subdivisions. The group is generated by $s_1 \equiv (bc)(1637)(2548)(df)$, and is of order 4.

The sets of transitive elements for II*C* are $a; b; c; d; e; f; g; 1, 3; 2, 4; 5, 8; 6, 7$; these with the classes separate the system into 23 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, and is of order 2.

The sets of transitive elements for II*D* are $a; b; c; d; e; f; g; 1, 3; 2, 4; 5, 8; 6, 7$; these with the classes separate the system into 23 non-permutable subdivisions. The group is generated by $s \equiv (13)(24)(58)(67)$, and is of order 2.

The two systems II*C and* II*D are distinct, since the number of triples appearing under each index is not the same for the two systems; but the group is identical for the two systems.* This is another example of the fact that two distinct systems may have the same group.

The sets of transitive elements for II*E* are $a; b; c; d, f, 6, 7; e, g, 5, 8; 1, 3; 2, 4$; these with the classes separate the system into 12 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (d7f6)(e5g8)(13)(24)$, and is of order 8.

* For the sake of brevity the work involved in the determination of the indices for every system is suppressed, and a brief outline of the process employed in obtaining the groups is given only for the system I*A*. A fuller elaboration is contained in a manuscript volume deposited in the Vassar College Library.

The sets of transitive elements for IIIF are a; b, c; d; f; e, g; 1, 3, 6, 7; 2, 4, 5, 8; these with the classes separate the system into 15 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (bc)(eg)(17)(28)(36)(45)$, and is of order 4.

Seven-by-four array with 14 quadrangular arrays

12. We construct next a seven-by-four array containing 14 quadrangular arrays, the greatest possible number, and we apply to this new array the re-heading process. The partial indices show 4 types of heads and we obtain the following systems:

	a	b	c	d	e	f	g	Number of triples with indices			
III	12	13	14	15	16	17	18				
	34	24	23	26	25	28	27				
	56	58	57	38	37	36	35				
	78	67	68	47	48	45	46	1^{12}	$1^8 2^2$	$1^6 3^2$	$1^4 2^4$
A	abc	ade	afg	bdg	bef	cdf	ceg	35			
B	abc	ade	afg	bdf	beg	cdg	cef	7	24		4
C	abc	adg	aef	bdf	beg	cde	cfg	1	12	16	6
D	abd	acg	aef	bcf	beg	cde	dfg			28	7

The system IIIA contains one set of transitive elements a, b, c, d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8, and the thirty-five triples belong to one class.

The analysis of the Kirkman system by the method of sequences shows the index of each triple to be 1^{12}, and the system IIIA is congruent to the Kirkman system. The group for the system IIIA is by far the most extensive of the 24 groups determined in this paper, and belongs to the earliest known Δ_{15}.

The group for IIIA is generated by $s_1 \equiv (12)(34)(56)(78)$, $s_2 \equiv (13)(24)(58)(67)$, $s_3 \equiv (15)(26)(38)(47)$, $s_4 \equiv (acb)(efg)(243)(678)$, $s_5 \equiv (bc)(fg)(34)(78)$, $s_6 \equiv (bc)(dfeg)(34)(5768)$, $s_7 \equiv (adgfceb)(1347286)$, $s_8 \equiv (ad7gfc5368b1e42)$, and is of order $8!/2$.*

The sets of transitive elements for IIIB are a; b, c, d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group for the system is generated by $s_1 \equiv (beg)(cdf)(12)(358467)$, $s_2 \equiv (df)(eg)(57)(68)$, $s_3 \equiv (15)(26)(38)(47)$, $s_4 \equiv (de)(fg)(56)(78)$, and is of order 96.

The sets of transitive elements for IIIC are a, b, c; d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group for the system IIIC is generated by $s_1 \equiv (abc)(dfg)(234)(578)$, $s_2 \equiv (16)(25)(37)(48)$, $s_3 \equiv (12)(34)(56)(78)$, $s_4 \equiv (dg)(ef)(58)(67)$, and is of order 96. The groups for the systems IIIB and IIIC are of the same order 96, but the sets of transitive elements for the two groups differ, hence the groups are distinct.

* C. Jordan (*Traité des substitutions et des équations algébriques* (1870), p. 380) states that the group for the Kirkman system is of order $8!/2$, which agrees with the result obtained by the sequence method of analysis.

The sets of transitive elements for III*D* are a, b, c, d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group is generated by $s_1 \equiv (acg)\,(bef)\,(248)\,(367)$, $s_2 \equiv (15)\,(26)\,(38)\,(47)$, $s_3 \equiv (abedgcf)\,(3658472)$, $s_4 \equiv (13)\,(24)\,(58)\,(67)$, and is of order 168.

Seven-by-four array with six quadrangular arrays

13. We construct finally a seven-by-four array containing 6 quadrangular arrays, the only remaining number (cf. §11) and we obtain the 4 following systems:

	a	*b*	*c*	*d*	*e*	*f*	*g*
IV	87	86	85	84	83	82	81
	65	75	76	73	74	71	72
	43	41	42	62	61	64	63
	21	23	31	15	25	53	54
A	*bdf*	*abc*	*beg*	*adg*	*cde*	*afe*	*cfg*
B	*bdf*	*abe*	*bcg*	*adg*	*cde*	*acf*	*efg*
C	*bdf*	*abe*	*cef*	*bcg*	*acd*	*deg*	*afg*
D	*bdf*	*abc*	*beg*	*ade*	*cdg*	*afg*	*ecf*

IV	1^{12}	$1^8 2^2$	$1^6 24$	$1^6 3^2$	$1^5 2^2 3$	$1^4 2^4$	$1^4 26$	$1^4 4^2$	$1^3 2^2 3^2$
A	1	8			16	10			
B		1		4	8	6	8	2	6
C	1	10	4		8	6		2	4
D	7	24				4			

The systems IV*C* and IV*D* show the same indices as the systems II*E* and III*B*, respectively, and are congruent to them.

The sets of transitive elements for IV*A* are $a, b; c; d, e, f, g$, 1, 2, 3, 4; 5, 6, 7, 8; and the system contains 7 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (ab)\,(d1)\,(e3)\,(f2)\,(g4)\,(67)$, $s_2 \equiv (ab)\,(d2f1)\,(e4g3)\,(5687)$, $s_3 \equiv (13)\,(24)\,(57)\,(68)$, and is of order 32.

The systems of transitive elements for IV*B* are $a; b; c; d; e; f; g$; 1, 2, 3, 4; 5, 6, 7, 8; and the system separates into 17 non-permutable subdivisions. The group is generated by $s_1 \equiv (12)\,(34)\,(56)\,(78)$, $s_2 \equiv (14)\,(23)\,(58)\,(67)$, and is of order 4.

Formation of new systems by varying the number of partial quadrangular arrays

14. The three sets of pairs 12, 34; 13, 24; 14, 23 entering the seven-by-four array in 3 columns form a quadrangular array which has been denoted by the symbol $\left[\begin{smallmatrix}12\\34\end{smallmatrix}\right]$. Two of these sets of pairs occurring in the seven-by-four array in 2 columns form a *partial* quadrangular array, and may be denoted by the symbol $\left\{\begin{smallmatrix}12\\34\end{smallmatrix}\right\}$. A partial quadrangular array considered with respect to the two elements which head the columns in which the two sets of pairs enter, is designated in a recent paper as an "interlacing"* of the two ele-

* F. N. Cole, *The triad systems of thirteen elements.* These Transactions, vol. 14 (1913), pp. 1–5.

ments which head the columns. Non-congruent triple-systems may be formed by keeping constant the number of quadrangular arrays in the seven-by-four array, and varying the number of partial quadrangular arrays.

The seven-by-four array II contains two quadrangular arrays $\left[\begin{smallmatrix}12\\34\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}56\\78\end{smallmatrix}\right]$ and eight partial quadrangular arrays $\left\{\begin{smallmatrix}12\\56\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}12\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\58\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\67\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}24\\58\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}24\\67\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}34\\56\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}34\\78\end{smallmatrix}\right\}$. The seven-by-four array V, given below, contains the same number of quadrangular arrays, $\left[\begin{smallmatrix}12\\58\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}34\\67\end{smallmatrix}\right]$, but a different number of partial quadrangular arrays, the twelve $\left\{\begin{smallmatrix}12\\46\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}12\\37\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\48\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\56\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}14\\57\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}16\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}23\\45\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}23\\68\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}24\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}25\\67\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}35\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}45\\68\end{smallmatrix}\right\}$. The re-heading process applied to this seven-by-four array shows six types of heads.

	a	*b*	*c*	*d*	*e*	*f*	*g*	Number of triples with partial indices			
V	12	15	18	14	17	13	16				
	58	28	25	26	23	27	24				
	34	37	36	38	45	48	35				
	67	46	47	57	68	56	78	1^8	$1^4 2^2$	2^4	4^2
A	*abc*	*ade*	*afg*	*bdf*	*beg*	*cdg*	*cef*	1		6	
B	*abd*	*ace*	*afg*	*bcf*	*beg*	*cdg*	*def*			4	3
C	*abd*	*ace*	*afg*	*bcg*	*bef*	*cdf*	*deg*		2	2	3
D	*abd*	*acf*	*aeg*	*bcg*	*bef*	*cde*	*dfg*		3	1	3
E	*abc*	*ade*	*afg*	*bdg*	*bef*	*cdf*	*ceg*	1	4	2	
F	*abc*	*adf*	*aeg*	*bdg*	*bef*	*cde*	*cfg*	1	6		

A determination of the indices of the 6 systems with fundamental seven-by-four array V gives the following result:

V	1^{12}	$1^8 2^2$	$1^6 3^2$	$1^5 2^2 3$	$1^4 2^4$	$1^4 35$	$1^4 4^2$	$1^3 234$	$1^3 3^3$	$1^2 2^2 3^2$
A	1				18				16	
B					4		3	12	4	12
C		2		4	6	8	3	4		8
D		3	4		7	12	3			6
E	1	8		16	10					
F	1	12	16		6					

The triple-systems V*E* and V*F* show the same sets of indices as IV*A* and III*C*, respectively, and are congruent to them. The analysis follows for the 4 systems V*A*, V*B*, V*C*, V*D*, which are distinct from the preceding 15 triple-systems.

The sets of transitive elements for V*A* are $a, b, c; d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8$. The group is generated by $s_1 \equiv (abc)\,(efg)\,(258)\,(367)$, $s_2 \equiv (bc)\,(d1)\,(e2)\,(f8)\,(g5)\,(67)$, $s_3 \equiv (d14)\,(e23)\,(f56)\,(g87)$, $s_4 \equiv (acb)\,(dfg)\,(182)\,(376)$, and is of order 288.

The sets of transitive elements for V*B* are $a, b, c; d, e, f; g; 1, 2, 3, 4, 5, 6, 7, 8$; these with the classes separate the system into 6 non-permutable subdivisions. The group is generated by $s_1 \equiv (ab)\,(ef)\,(1327)\,(4865)$, $s_2 \equiv (ab)\,(ef)\,(14)\,(26)\,(35)\,(78)$, $s_3 \equiv (acb)\,(def)\,(182)\,(467)$, and is of order 24.

The sets of transitive elements for VC are a; b, c; d, e; f; g; 1, 2, 3, 4, 5, 6, 7, 8; these with the classes separate the system into 11 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (bc)\,(de)\,(16)\,(27)\,(35)\,(48)$, $s_2 \equiv (bc)\,(de)\,(1457)\,(2386)$, and is of order 8.

The sets of transitive elements for VD are a, b, c; d, f, g; e; 1, 2, 5, 8; 3, 4, 6, 7; these with the classes separate the system into 7 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (12)\,(37)\,(46)\,(58)$, $s_2 \equiv (18)\,(25)\,(34)\,(67)$, $s_3 \equiv (acb)\,(dfg)\,(125)\,(347)$, and is of order 12.

Analysis of triple-systems formed by other methods

15. A few of the well-known systems Δ_{15} are now examined, in order to determine, if possible, systems incongruent to the 19 already given.

(i) *Analysis of the Carpmael systems.* – The indices for the systems given by Carpmael* show two non-congruent systems. One of these yields a set of indices identical with that of VA; hence this system is omitted from further examination. The other Carpmael system yields a new set of indices; therefore we determine in this system a Δ_7, arrange the system in the form of a seven-by-four array, and apply the re-heading process, thus obtaining four systems incongruent to the preceding 19, VIA, VIB, VIC, and VID of the following table:

	a	b	c	d	e	f	g							
VIA	12	17	16	15	18	14	13	*abe*	*acf*	*adg*	*bdf*	*cde*	*efg*	*bcg*
VIB	34	23	24	28	27	25	26	*abc*	*adg*	*aef*	*bde*	*bfg*	*cdf*	*ceg*
VIC	58	48	38	36	35	37	45	*abd*	*acg*	*aef*	*bce*	*bfg*	*cdf*	*deg*
VID	67	56	57	47	46	68	78	*abg*	*ace*	*adf*	*bcd*	*bef*	*cfg*	*deg*

A comparison of the two seven-by-four arrays I and VI shows that neither contains any quadrangular array, but that while I contains no partial quadrangular array, VI contains the following six $\{^{12}_{58}\}$, $\{^{14}_{68}\}$, $\{^{15}_{67}\}$, $\{^{23}_{48}\}$, $\{^{23}_{57}\}$, and $\{^{34}_{67}\}$. Hence the two sets of systems I and VI may be formed by constructing seven-by-four arrays, which contain a fixed number (zero) of quadrangular arrays and a variable number of partial quadrangular arrays. The analysis and the groups for the system VI are given below.

	$1^8\,2^8$	$1^6\,24$	$1^6\,2^2\,3$	$1^5\,7$	$1^4\,2^4$	$1^4\,35$	$1^4\,4^2$	$1^3\,2^2\,5$	$1^3\,234$	$1^2\,2^3\,4$	$1^2\,2^2\,3^2$	$1^3\,5^2$	1236	1245	$13^2\,5$	$2^3\,6$	2^8
VIA				4	3			6					12		6		4
VIB	1	2		2	1	2	1	3	2	2	8	2	2	4	1	2	
VIC		3		1			3			3	6	9	6	3			1
VID	3	3	6	1		3		3		3	6	6					1

* *Some solutions of Kirkman's 15-school-girl problem*, Proceedings of the London Mathematical Society, vol. 12 (1881), pp. 148–156.

The sets of transitive elements for VIA are $a, b, c, d, e, f; g; 1, 2, 4, 7; 3, 5, 6, 8$. The group is generated by $s_1 \equiv (abe)\,(cfd)\,(172)(386)$, $s_2 \equiv (ad)\,(bc)\,(14)\,(27)\,(35)\,(68)$, $s_3 \equiv (bc)\,(ef)\,(12)\,(36)(47)\,(58)$, and is of order 12.

The sets of transitive elements for VIB are $a; b, c; d; e, f; g; 1, 2; 3, 6; 4, 7; 5, 8$. These with the classes separate the system into 21 non-permutable subdivisions. The group is of order 2 and is generated by $s \equiv (bc)\,(ef)\,(12)\,(36)\,(47)\,(58)$.

The sets of transitive elements for VIC are $a, b, e; c, d, f; g; 1, 2, 7; 4; 5; 3, 6, 8$. These with the classes separate the system into 13 non-permutable subdivisions. The group for the system is generated by $s \equiv (aeb)\,(cdf)\,(127)\,(368)$, and is of order 3.

The sets of transitive elements for VID are $a, c, e; b, d, f; g; 1, 4, 7; 2; 6; 3, 5, 8$. These with the classes separate the system into 13 non-permutable subdivisions. The group for the system is generated by $s \equiv (ace)\,(bfd)\,(147)\,(358)$, and is of order 3.

A comparison of the systems IB, IC, VIC, VID, shows different sets of indices but identical groups. *Here then are four incongruent systems with the same group* (cf. §8 and §9). *Therefore it is evident that the identity of the groups, though a necessary condition for the congruency of two triple-systems, is not sufficient.*

(ii) *Analysis of the Heffter systems.* — The Heffter* method of constructing triple-systems of $6m + 3$ elements furnishes two systems for $6m + 3 = 15$. The first of these contains 15 systems Δ_7, the index of each triple is 1^{12}, and the system shows the same set of indices as the system here denoted by IIIA. The second system contains no Δ_7, and is therefore distinct from the preceding 23 systems, each of which contains at least one Δ_7. This second system is now denoted by VII. Its analysis is shown in the following table:

Classification of VII by indices

4, 8		3, 9									
*af*4	*d*27	*a*58	*ag*7	*bc*7	*b*24	*ce*4	*df*5	*eg*6	*f*17	*g*28	367
*bg*5	*e*38	*ade*	*ac*2	*bef*	*cd*8	*c*35	*d*46	*e*57	*f*68	145	478
*c*16		*ab*6	*a*13	*b*18	*cfg*	*dg*1	*e*12	*f*23	*g*34	256	*bd*3

The system contains 1 set of transitive elements: $a, b, c, d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8$, and separates into 2 non-permutable subdivisions. The group is generated by $s_1 \equiv (ad63)\,(c8f2)\,(e471)$, $s_2 \equiv (abcdefg12345678)$, and is of order 60.

(iii) Netto† describes two methods for constructing triple-systems of 15

* L. Heffter, *Ueber Tripelsysteme*, Mathematische Annalen, vol. 49 (1897), pp. 101–112.

† L. c.

elements. The analysis of the two systems by the method of sequences shows that the index of every triple in each system is 1^{12} and the two systems are of the type here designated by IIIA.

(iv) E. H. Moore* gives different methods for constructing triple-systems. A system formed by one of these methods was found to be of the type IIIA. Under the re-heading process a system formed by a second method yielded 4 systems congruent, respectively, to the systems here designated as IIE, IIIB, IIIC, IVA.

16. We have shown for the two fundamental seven-by-four arrays containing, respectively, zero and two quadrangular arrays, that non-congruent systems may be formed by varying the number of partial quadrangular arrays. It seems possible that additional systems Δ_{15}, each containing at least one Δ_7, might be formed by the same method from these two arrays, and also from the two fundamental arrays containing, respectively, six and fourteen quadrangular arrays. The purpose of this investigation is primarily to exhibit a new method of comparison for triple-systems, and while several systems Δ_{15} not hitherto constructed have been obtained, no exhaustive determination of all possible systems Δ_{15} has been undertaken.

The method of comparison here investigated is applicable to any system Δ_{2t+1} by merely arranging the system in the form of a $2t + 1$-by-t array and determining the indices for this array. In constructing systems Δ_{15} which contain no Δ_7, the number of partial quadrangular arrays admitted into a fifteen-by-seven array must be considered, but for systems Δ_{15} which contain one or more systems Δ_7, the number of complete and partial quadrangular arrays admitted into a seven-by-four array is to be considered. Exactly similar conditions exist in the case of systems Δ_{2t+1}. In constructing systems Δ_{2t+1} which contain no Δ_t, the consideration of a $2t + 1$-by-t array is necessary, but for systems Δ_{2t+1} which contain one or more systems Δ_t, the consideration of a t-by-$\frac{1}{2}(t + 1)$ array is sufficient.

April, 1913

* L. c.

AN EXISTENCE THEOREM FOR A CERTAIN DIFFERENTIAL EQUATION OF THE nTH ORDER*

BY

WILLIAM RAYMOND LONGLEY

1. The usual method for investigating the solutions of a differential equation of the nth order in the neighborhood of a singular point has been to reduce the problem to a system of n equations of the first order. This procedure was adopted by Poincaré† and others for the treatment of equations of a general form, and the results obtained have been summarized and simplified by Dulac.‡ No explicit statement has been made concerning an equation of the form

$$\frac{d^n y}{dx^n} = \frac{N}{D} = \frac{ax + by + \cdots}{\alpha x + \beta y + \cdots}, \tag{1}$$

where N and D are convergent power series in x and y vanishing for $x = 0$, $y = 0$, but an application of the general theory yields the following information concerning the integrals in the neighborhood of the values $x = 0, y = 0$.

Equation (1) may be reduced to the system

$$\frac{dy}{dx} = c + y_1, \qquad \frac{dy_1}{dx} = c_2 + y_2, \qquad \cdots, \qquad \frac{dy_{n-1}}{dx} = \frac{N}{D}, \tag{2}$$

where $c, c_2, \cdots, c_{n-1}$ are arbitrary finite constants and the initial values of the variables are all zero. If $\lambda = \alpha + c\beta$ is not zero then the variables can be expressed as power series in terms of a single quantity z; that is,

$$x = x(z), \qquad y = y(z), \qquad \cdots, \qquad y_{n-1} = y_{n-1}(z), \tag{3}$$

such that if z satisfies an equation

$$\frac{dz}{dt} = \lambda z + z^2 P, \tag{4}$$

then the functions (3) will satisfy the system (2). The quantity P is a power

* Presented to the Society, December 30, 1913.

† Journal de mathématiques, ser. 4, vol. 1 (1885), p. 167, and vol. 2 (1886), p. 151.

‡ Bulletin de la société mathématique de France, vol. 40 (1912), p. 324.

series in z whose coefficients depend upon the system (2). The solution z of equation (4) can be expressed as a power series in $\theta = ke^{\lambda t}$, k being an arbitrary constant. After substitution of this expression in the system (3), θ can be found in terms of x from the first equation and, when this value is put in the second, y is obtained as a function of x which satisfies equation (1). This solution involves the $n-1$ finite constants $c, c_2, \cdots, c_{n-1}$, which are arbitrary except for the condition $\alpha + c\beta \neq 0$. The exact form of the solution is not given and its construction by this general method would involve a number of transformations and considerable calculation.

The object of the present paper is to obtain this result directly in explicit form and to derive results in some cases for which $\alpha + c\beta = 0$.

2. By setting $y = x(c + v)$, where v is to be determined as a function of x vanishing with x, and c is a constant, equation (1) becomes

$$x\frac{d^n v}{dx^n} + n\frac{d^{n-1} v}{dx^{n-1}} = \frac{a + b(c+v) + xN'}{\alpha + \beta(c+v) + xD'}.$$

If $\alpha + c\beta \neq 0$ the second member can be expanded as a power series in x and v, and the equation takes the form

$$x\frac{d^n v}{dx^n} + n\frac{d^{n-1} v}{dx^{n-1}} = a_{00} + a_{10}x + a_{01}v + \cdots + a_{ij}x^i v^j + \cdots. \tag{5}$$

Suppose a solution is assumed in the form

$$v = c_2 x + c_3 x^2 + \cdots + c_{n-1}x^{n-2} + c_n x^{n-1} + \cdots + c_{m+1}x^m + \cdots. \tag{6}$$

On substituting this value of v in equation (5) and equating the coefficients of corresponding powers of x, it follows that $c_2, c_3, \cdots, c_{n-1}$ may be chosen arbitrarily, while the remaining coefficients of the series (6) are uniquely determined by

$$\begin{aligned} &\underline{n}\, c_n = a_{00}, \\ &\cdots\cdots\cdots, \\ &\{m(m-1)\cdots(m-n+1) \\ &\qquad + nm(m-1)\cdots(m-n+2)\}c_{m+1} \\ &\qquad\qquad = F_{m+1}(a_{ij}, c_2, \cdots, c_{n-1}), \end{aligned} \tag{7}$$

where F_{m+1} is a rational integral function of $c_2, \cdots, c_{n-1}$ and the coefficients $a_{ij}\,(i + j < m - n)$.

The convergence of the series (6) can be demonstrated by comparing it with the solution of

$$\frac{d^{n-1} u}{dx^{n-1}} = a'_{00} + a'_{10}x + a'_{01}u + \cdots + a'_{ij}x^i u^j + \cdots, \tag{8}$$

the coefficients a'_{ij} being positive constants equal to the absolute values of the a_{ij}. By Cauchy's fundamental theorem it is known that this equation has a solution for u as a power series in x which is convergent for values of x whose moduli are sufficiently small. The solution is

$$(9) \qquad u = \alpha_2 x + \cdots + \alpha_{n-1} x^{n-2} + \alpha_n x^{n-1} + \cdots + \alpha_{m+1} x^m + \cdots,$$

where $\alpha_2, \cdots, \alpha_{n-1}$ are arbitrary. The remaining coefficients are determined by

$$(10) \qquad \begin{aligned} &\lfloor n-1 \, \alpha_n = a'_{00}, \\ &\cdots\cdots\cdots\cdots, \\ &m(m-1)\cdots(m-n+2)\,\alpha_{m+1} = F'_{m+1}(a'_{ij}, \alpha_2, \cdots, \alpha_{n-1}). \end{aligned}$$

The function F'_{m+1} in equations (10) is a rational integral function of the same form as F_{m+1} in equations (7), the coefficients being all positive and greater than the absolute values of the corresponding coefficients in F_{m+1}. It follows then that the absolute value of c_{m+1} is less than α_{m+1} for every m provided $\alpha_2 = c_2, \cdots, \alpha_{n-1} = c_{n-1}$. Hence the series (6) converges for at least the same range of values as the series (9).

The solution of equation (1) is now given in the explicit form

$$y = cx + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + \cdots + c_m x^m + \cdots,$$

where $c, c_2, \cdots, c_{n-1}$ are finite constants, arbitrary except for the condition $\alpha + c\beta \neq 0$. The remaining coefficients are uniquely determined in terms of the arbitrary constants and the coefficients in the differential equation. This statement contains the information which may be derived from the general theory as summarized by Dulac.

3. The preceding method may be applied without alteration to the case when in equation (1) the lowest terms in N are of degree p, and the lowest terms in D are of degree q if $p \geqq q$. If the terms of lowest degree in D are

$$D_0 x^q + D_1 x^{q-1} y + \cdots + D_q y^q,$$

the assumption concerning the constant c is

$$D_0 + D_1 c + \cdots + D_q c^q \neq 0.$$

Geometrically this assumption is expressed by saying that the integral curve is not tangent at the origin to a branch of the curve $D = 0$.

The existence theorem may now be stated in the following form:

THEOREM. *Given the differential equation*

$$(E) \qquad \frac{d^n y}{dx^n} = \frac{N}{D} = \frac{N_0 x^p + N_1 x^{p-1} y + \cdots + N_p y^p + \cdots}{D_0 x^q + D_1 x^{q-1} y + \cdots + D_q y^q + \cdots}$$

in which N and D are convergent series in positive integral powers of x and y

and $p \geqq q$. *If*

(*C*) $$D_0 + D_1 c + \cdots + D_q c^q \neq 0,$$

then there exists a solution expressible in the form

(*S*) $$y = cx + \sum_{i=2}^{\infty} c_i x^i,$$

which is convergent for values of x *whose moduli are sufficiently small, and the coefficients of the first* $n - 1$ *powers of* x *are arbitrary except for the condition* (*C*).

The fundamental theorem of Cauchy applies to an equation of the form

(11) $$\frac{d^n y}{dx^n} = P,$$

in which P is a series in positive integral powers of x and y, and asserts the existence of a solution of the form (*S*) in which the coefficients of the first $n - 1$ powers of x are entirely arbitrary. The difference in the results may be stated geometrically by saying that for equation (11) the regular integral curves through the origin may go in *any* direction, while for the equation (*E*) they may go in any direction not tangent to a branch of the curve $D = 0$ (excluding always the direction of the y-axis).

4. We now return to equation (1) in which the numerator and denominator contain terms of the first degree and inquire for integral curves tangent at the origin to the curve $D = 0$. We ask if the equation can be satisfied by expressing y as a series in positive integral powers of x^σ, where σ is a positive constant. Since $dy/dx = -\alpha/\beta = c$ for $x = 0$, we make the substitution $y = -\alpha x/\beta + v$, and v is to be determined as a series in x^σ containing x^k as a factor, where k is a positive number greater than unity, and satisfying the equation

(12) $$\frac{d^n v}{dx^n} = \frac{Ax + bv + P_2}{\beta v + Q_2}.$$

In this equation P_2 and Q_2 contain no first degree terms, and $\beta A = a\beta - b\alpha$. Let the term in Q_2 of lowest degree independent of v be x^r. Then r is an integer greater than unity. Suppose that A is not zero, which means that the curves $N = 0$ and $D = 0$ are not tangent at the origin.

Let a solution of equation (12) be assumed in the form

(13) $$v = lx^k + mx^{k+\sigma} + \cdots \qquad (l \neq 0).$$

After multiplying equation (12) by the denominator of the second member and substituting the value of v, the index of the lowest power of x must be the same on each side of the equation. Two cases are to be considered.

I. Suppose k is not an integer less than n. If k is not greater than r, the lowest term in the left member is of degree $2k - n$, and in the right member is of the first degree. Hence $k = (n + 1)/2$, which is possible only if n is even. If k is greater than r, the index equation is $k - n + r = 1$, which is contrary to the assumption that k is not an integer less than n.

II. Suppose k is an integer less than n. If σ is not an integer, the index equation is $2k + \sigma - n = 1$ or $k + \sigma + r - n = 1$, which contradicts the assumption that σ is not an integer. If σ is an integer it may be taken as unity without loss of generality. The lowest power of x in $d^n v/dx^n$ will be x^q, where q is zero or a positive integer, and the index equation becomes $k + q = 1$, which contradicts the assumption that k is greater than unity, or $r + q = 1$, which is impossible.

The preceding argument shows that equation (1) may admit a solution for y expressible as a positive power series in x^σ $(\sigma > 0)$ such that $dy/dx = -\alpha/\beta$ when $x = 0$, only if n is even. If such a series exists the first two terms will be

$$(N) \qquad y = -\frac{\alpha}{\beta}x + lx^{(n+1)/2} + \cdots.$$

5. We shall consider in detail the case when $n = 2$, and show that equation (1) admits a solution for y as a power series in $x^{\frac{1}{2}}$. On introducing ξ as the independent variable by the relation $x = \xi^2$, equation (1) becomes

$$(14) \qquad \frac{1}{4\xi^3}\left\{\xi\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\right\} = \frac{a\xi^2 + by + \cdots}{\alpha\xi^2 + \beta y + \cdots}.$$

Following the condition (N) it is convenient to set

$$y = -\frac{\alpha}{\beta}\xi^2 + \xi^3(l + v),$$

where v is to be determined as a power series in ξ vanishing with ξ. Then equation (14) takes the form

$$(15) \qquad \xi^2\frac{d^2v}{d\xi^2} + 5\xi\frac{dv}{d\xi} + 3(l + v) = \frac{4A + \xi P_0}{\beta l + \beta v + \xi Q_0},$$

in which P_0 and Q_0 are series in ξ and v. The constant l is determined by setting $\xi = 0$, $v = 0$ which gives $3\beta l^2 = 4A$. The second member of equation (15) can be expanded as a power series in ξ and v, and the equation becomes

$$(16) \qquad \xi^2\frac{d^2v}{d\xi^2} + 5\xi\frac{dv}{d\xi} + 6v = A\xi + \cdots + A_{ij}\,\xi^i v^j + \cdots \qquad (i + j > 1).$$

Let a solution be assumed in the form

$$(17) \qquad v = a_1\xi + a_2\xi^2 + \cdots + a_n\xi^n + \cdots.$$

The equations for the determination of the coefficients are

$$(18) \qquad (5+6)a_1 = A, \quad \cdots, \quad \{n(n-1)+5n+6\}a_n = F_n.$$

The quantities F_n are rational integral functions of $A_{ij}\,(i+j \leqq n)$, and the coefficients in equation (17) are uniquely determined.

The convergence of the series (17) can be demonstrated by comparing it with the solution of

$$(19) \qquad 5\xi\frac{du}{d\xi} + 6u = A'\xi + \cdots + A'_{ij}\,\xi^i w^j + \cdots \qquad (i+j>1),$$

the coefficients A'_{ij} being positive constants equal to the absolute values of the A_{ij}. If a solution is assumed in the form

$$(20) \qquad u = \alpha_1\xi + \cdots + \alpha_n\xi^n + \cdots,$$

the equations for the determination of the coefficients are

$$(21) \qquad (5+6)\alpha_1 = A', \quad \cdots, \quad (5n+6)\alpha_n = F'_n.$$

In equations (21), F'_n is a rational integral function of the same form as F_n in equations (18), the coefficients being all positive and greater than the absolute values of the corresponding coefficients in F_n. Evidently the absolute value of a_n is less than α_n for every n. Now equation (19) is the well-known equation of Briot and Bouquet* which admits a solution of the form (20) if the coefficient of u (in this case 6/5) is not a negative integer. The series (20) is therefore known to be convergent for values of x whose moduli are sufficiently small. Consequently the series (17) is convergent for at least the same range of values.

The conclusion is that if $a\beta - b\alpha \neq 0$, then equation (1), in which $n = 2$, admits a solution of the form

$$y = -\frac{\alpha}{\beta}x + x^{\frac{1}{2}}P,$$

where P is a power series in $x^{\frac{1}{2}}$.

When this result is combined with that of the theorem above, we can make the following geometric statement concerning a differential equation of the form (1) and of the second order. If the origin is an ordinary point on each of the curves $N = 0$ and $D = 0$, and if these curves have distinct tangents at the origin, then, in every direction through the origin (except, perhaps, tangent to the y-axis) there passes one integral curve. These integral curves are all regular except the one tangent to the curve $D = 0$. This one has a branch point at the origin.

It has been said above that for n even the equation (1) may possibly admit a

*Journal de l'école polytechnique, vol. 21 (1856), p. 161.

solution of the form (N). For n greater than 2 the search for such solutions involves the terms in D of order higher than the first and the investigation will not be undertaken in this paper.

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